P VERSUS NP AND GEOMETRY

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ABSTRACT. I describe three geometric approaches to resolving variants of P v. NP, present several results that illustrate the role of group actions in complexity theory, and make a first step towards completely geometric definitions of complexity classes.

1. INTRODUCTION

Roughly speaking, a *problem* in complexity theory is a class of expressions to evaluate (e.g. count the number of four colorings of a planar graph). An *instance* of a problem is a particular member of the class (e.g. count the number of four colorings of the complete graph with four vertices). **P** is the class of problems that admit an algorithm that solves any instance of it in a number of steps that depends polynomialy on the size of the input data. One says that such problems "admit a polynomial time solution". **NP** is the class of problems where a proposed solution to an instance can be positively checked in polynomial time. The famous Cook's hypothesis is $\mathbf{P} \neq \mathbf{NP}$.

I will be concerned with two types of evaluations in this article, here is the first: For each n, let V_n be a complex vector space and assume $\dim(V_n)$ grows exponentially with n. It is known that the pairing

$$V_n \times V_n^* \to \mathbb{C}$$
$$(v, \alpha) \mapsto \langle \alpha, v \rangle$$

of the vector space with its dual requires on the order of $\dim(V_n)$ arithmetic operations to perform. However if V_n has additional structure and α, v are in "special position" with respect to this structure, the pairing can be evaluated faster. A trivial example would be if V_n were equipped with a basis and v was restricted to be a linear combination of only the first few basis vectors. I will be concerned with more subtle examples such as the following: let $V_n = \Lambda^k \mathbb{C}^n$, then inside V_n are the decomposable vectors (the cone over the Grassmannian $G(k, \mathbb{C}^n)$) and if α, v are decomposable we show (Equation (3.1.1)) that the pairing $\langle \alpha, v \rangle$ can be evaluated in polynomial time in n. From a geometric perspective, this is one of the key ingredients to L. Valiant's *holographic algorithms* discussed in §4. For n large, the codimension of the Grassmannian is huge, so it would seem highly unlikely that any interesting problem could have α, v so special. However small Grassmannians are of small codimension. This leads to the second key ingredient to holographic algorithms. On the geometric side, if $[v_1] \in G(k_1, W_1)$ and $[v_2] \in G(k_2, W_2)$, then $[v_1 \otimes v_2] \in G(k_1 k_2, W_1 \otimes W_2)$. Thus if our vectors can be thought of as being built out of vectors in smaller spaces, there is a much better chance of success. Due to the nature of problems in complexity theory, this is exactly what occurs. The third key ingredient is that there is some flexibility in how the small vector spaces are equipped with the additional structure, and I show (Theorem 4.3.2) that even for **NP**-complete problems there is sufficient flexibility to allow everything to work up to this point. The difficulty occurs when one tries to

glue together the small vector spaces compatibly for both V_n and V_n^*, although even here, it is surprising (at least to me) that the "only" problem that can occur is one of signs, see §4.4.

The second type of evaluation I will be concerned with is that of sequences of (homogeneous) polynomials, $p_n \in S^{d(n)}\mathbb{C}^{v(n)}$, where the degee $d(n)$ and the number of variables $v(n)$ are required to grow at most polynomially with n. A generic such sequence is known to require an exponential (in n) number of arithmetic operations to evaluate and we are interested in characterizing the sequences where the evaluation can be done quickly. Again there are sequences such as $p_n = x_1^{d(n)} + \cdots + x_{v(n)}^{d(n)}$ where it is trivial to see that there is a polynomial time evaluation, but there are other, more subtle examples, such as $\det_n \in S^n\mathbb{C}^{n^2}$ where the fast evaluation occurs thanks to a group action (Gaussian elimination, see §2.1).

From a geometer's perspective, it is more interesting to look at the zero sets of the polynomials, to get sequences of hypersurfaces in projective spaces. Similar to the situation above regarding signs, if one changes the signs in the expression of the determinant, e.g., to all plus signs to obtain the permanant, one arrives at a **VNP**-hard sequence, where **VNP** is Valiant's algebraic analogue of **NP**, see §6 for a definition.

Problem: Determine geometric properties of sequences of hypersurfaces such that their defining equations admit polynomial time evaluations.

First steps towards resolving this problem are taken in this paper. A second problem, which seems more difficult to me at this writing is:

Problem: Determine geometric properties of sequences of hypersurfaces such that their defining equations are in the class **VNP**.

A first observation is that if a polynomial is easy to evaluate, then any specialization of it is also easy to evaluate, or in other words the polynomial associated to any linear section of its zero set is also easy to evaluate. This leads to Valiant's conjecture that the permanent sequence (perm_m) cannot be realized as a linear projection of the determinant sequence $(\det_{n(m)})$ unless n grows faster than any polynomial (Conjecture 2.3.3). The best results on this conjecture so far are due to T. Mignon and N. Ressayre [23] who use local differential geometry. While the local differential geometry of the det_n-hypersurface is essentially understood (see Theorem 2.4.1), a major difficulty in continuing their program is to distinguish the local differential geometry of the perm$_m$-hypersurface from that of a generic hypersurface. Furthermore, the determinant hypersurface is so special it may be difficult to isolate exactly which of its properties are the key to it having a fast evaluation. Suggestions for overcoming this second difficulty are given in §7.2.

From the geometric point of view, a significant aesthetic improvement towards approaching Valiant's conjecture is the *Geometric complexity theory* (GCT) program proposed by K. Mulmuley and M. Sohoni. Instead of regarding the determinant itself, one considers its GL_{n^2}-orbit closure in $\mathbb{P}(S^n\mathbb{C}^{n^2})$ and similarly for the permanent. The problem becomes one to compare two algebraic varieties that are invariant under a group action. In §7.1 I briefly review the program and discuss ways of furthering the program, summarizing from [2]. Even with the GCT program, one still begins with the determinant and permanent, and it might be useful to consider other sequences as well, as discussed in §7.2.

1.1. **Overview.** In §2.1 I discuss Valiant's conjecture regarding the permanent as a projection of the determinant. There are two new results (Theorems 2.4.1 and 2.4.2) on the local differential geometry of the hypersurface $\{\det_n = 0\}$ relevant for complexity. In §3 I describe how the big cells in the Grassmannian (resp. the spinor variety) admit interpretations as the set of vectors of minors (resp. sub-Pfaffians) in preparation for §4, where I review the reformulation of holographic algorithms of [20] and point out a consequence that all problems in **NP** are "nearly"

holographic (Theorem 4.3.2). In §5 I generalize the results of §3 to all cominuscule varieties. In §6 I review the definitions of Valiant's complexity classes in preparation for §7 and §8. In §7 I very briefly review the Geometric Complexity Theory program of Mulmuley and Sohoni and mention ideas for furthering it. The examples up to this point indicate that sequences in **VP** that are not in **VP**$_e$ (and analogously for **P**) should have some kind of symmetry, but that symmetry could be hidden. A central goal of this paper is to try to formalize that idea. Another goal is to begin a discussion of how to have purely geometric definitions of complexity classes. In §8 this is accomplished for **VP**$_e$ (where *joins* and *multiplicative joins* play a role, the latter perhaps being defined here for the first time), and a first attempt is made to accomplish this for **VP**, using the idea of possibly hidden symmetries. Other than as noted above, the various sections can be read independently.

The results presented in this paper are preliminary but I hope they indicate some of the deep and beautiful connections between the **P** v.s **NP** problem and geometry. For connections with other areas of mathematics, see, e.g. [44].

2. Projecting the determinant to the permanent

2.1. **Complexity of** $(\det_n)$. For a vector space V, let $S^d V$ denote the space of homogeneous polynomials of degree d on the dual space V^*. Let $E, F = \mathbb{C}^n$, and let $E \otimes F$ denote the space of linear maps $E^* \to F$. The polynomial $\det_n \in \Lambda^n E \otimes \Lambda^n F \subset S^n(E \otimes F)$ is the unique up to scale (nonzero) element of the one-dimensional vector space $\Lambda^n E \otimes \Lambda^n F$. $\det_n$ is invariant under the action of $SL(E) \times SL(F)$, as $\det(axb) = \det(a)\det(x)\det(b)$. Fix bases in E, F, so we may identify $E \otimes F$ with the space of $n \times n$ matrices and $SL(E)$ as the subgroup of all $n \times n$ matrices with determinant one. If $x \in E^* \otimes F^*$ is expressed as a matrix, letting $\mathfrak{S}_n$ denote the permutation group on n elements, then

$$\det_n(x) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) x^1_{\sigma(1)}, ..., x^n_{\sigma(n)}.$$

In the naïve computation of $\det_n$, one uses $(n-1)(n!)$ multiplications and $n! - 1$ additions. If one uses Laplace expansions it lowers a little, but is still exponential. Nevertheless, one has the essentially classical:

Proposition 2.1.1. $(\det_n) \in$ **VP**. *More precisely,* $\det_n$ *can be evaluated by performing* $\mathcal{O}(n^4)$ *arithmetic operations.*

Fixing bases of E, F and identifying $E^* \otimes F^*$ with the space of $n \times n$ matrices, there are subspaces of $E^* \otimes F^*$ on which det can be evaluated by performing n arithmetic operations, for example the upper-triangular matrices which we will denote by $\mathfrak{b}$.

$\det_n$ is invariant under the action of the subgroup $U \subset SL(E)$ of all upper-triangular matrices with 1's on the diagonal as well as the group $\mathcal{W}$ of permutation matrices in $SL(E)$.

Proposition 2.1.1 essentially follows from:

Proposition 2.1.2 (Gaussian elimination)**.** *Notations as above, given $x \in E^* \otimes F^*$, there exists g in the group generated by U and W such that $g \cdot x \in \mathfrak{b}$. Such a g can be computed by performing a number of arithmetic operations that is polynomial in $n = \dim E$.*

proof of Prop. 2.1.1. For sufficently generic matrices the algorithm is clear. For an algorithm that works for arbitrary matrices, see [1]. □

2.2. **The permanent.** Define the permanent $\mathrm{perm}_n \in S^n(E \otimes F)$ to be the unique up to scale element of $S^n E \otimes S^n F \subset S^n(E \otimes F)$ invariant under the action of the diagonal matrices and permutation matrices acting on both the left and the right (i.e. the normalizers of the tori in $SL(E) \times SL(F)$). If $x \in E^* \otimes F^*$ is expressed as a matrix, then

$$\mathrm{perm}_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^1_{\sigma(1)}, ..., x^n_{\sigma(n)}.$$

2.3. **The permanent as a projection of the determinant.** A polynomial $p(y_1, ..., y_m)$ is a *projection* of $q(x_1, ..., x_n)$ if we can set $x_i = a^s_i y_s + c_i$ for constants a^s_i, c_i to obtain $p(y_1, ..., y_m) = q(a^s_1 y_s + c_1, ..., a^s_n y_s + c_n)$. Geometrically, if we homogenize the polynomials by adding variables y_0, x_0, we can study the zero sets in projective space. Then p is a projection of q iff $\mathrm{Zeros}(p) \subset \mathbb{CP}^m$ is a linear section of $\mathrm{Zeros}(q) \subset \mathbb{CP}^n$. This is because if we consider a projection map $V \to V/W$, then $(V/W)^* \simeq W^\perp \subset V^*$.

Theorem 2.3.1. *[Valiant][41] Every $f \in \mathbb{C}[x_1, ..., x_n]$ of expression size (see §6.1.4) u is both a projection of $\det_{u+3}$ and perm_{u+3}.*

In particular, any polynomial is the projection of some determinant.

Example 2.3.2. Let $f(x) = x_1 x_2 x_3 + x_4 x_5 x_6$, then

$$f(x) = \det \begin{pmatrix} 0 & x_1 & 0 & x_4 & 0 \\ 0 & 1 & x_2 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_5 \\ x_6 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conjecture 2.3.3 (Valiant)**.** *[33] Let $n(m)$ be the smallest integer such that perm_m can be realized as a projection of $\det_{n(m)}$. Then $n(m)$ grows faster than any polynomial in m.*

2.4. **Differential invariants of $\det_n$.** This subsection discusses preliminary results of work with D. The and L. Manivel.

Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be varieties such that there is a linear space $L \simeq \mathbb{P}^m \subset \mathbb{P}^n$ such that $Y = X \cap L$.

Say $y \in Y = X \cap L$. Then the differential invariants of X at y will project to the differential invariants of Y at y. A definition of differential invariants adequate for this discussion (assuming X, Y are hypersurfaces) is as follows: choose local coordinates $(x^1, ..., x^{n+1})$ for $\mathbb{P}^n$ at $x = (0, ..., 0)$ such that $T_x X = \langle \frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n} \rangle$ and expand out a Taylor series for X:

$$x^{n+1} = r^2_{i,j} x^i x^j + r^3_{i,j,k} x^i x^j x^k + \cdots$$

The zero set of $(r^2_{ij} dx^i \circ dx^j, ..., r^k_{i_1,...,i_k} dx^{i_1} \circ \cdots \circ dx^{i_k})$ in $\mathbb{P}T_x X$ is independent of choices. Write this zero set as $\mathrm{Zeros}(F_{2,x}(X), ..., F_{k,x}(X))$. I will refer to the polynomials $F_{\ell,x}(X)$ although they are not well defined individually. For more details see, e.g. [16, Chap. 3].

One says that X can approximate Y to k-th order at $x \in X$ mapping to $y \in Y$ if one can project the differential invariants to order k of X at x to those of Y at y.

In [23] it was shown that the determinant can approximate *any* polynomial to second order if $n \geq \frac{m^2}{2}$ and that perm_m is generic to order two, giving the lower bound $n(m) \geq \frac{m^2}{2}$. The

previous lower bound was $n(m) \geq \sqrt{2}m$ due to J. Cai [5] building on work of J. von zur Gathen [43].

One can ask what happens to higher orders.

If $X \subset \mathbb{P}V$ is a quasi-homogeneous variety, i.e., a group G acts linearly on V and $X = \overline{G \cdot [v]}$ for some $[v] \in \mathbb{P}V$, then $T_{[v]}X$ is a $\mathfrak{g}([v])$-module, where $\mathfrak{g}([v])$ denotes the Lie algebra of the stabilizer of $[v]$ in G.

Let $e^1, ..., e^n$ be a basis of E^* and $f^1, ..., f^n$ a basis of F^*, let $v = e^1 \otimes f^1 + \cdots + e^{n-1} \otimes f^{n-1}$, so $[v] \in \text{Zeros}(\det_n)$ and $\text{Zeros}(\det_n) = \overline{SL(E) \times SL(F) \cdot [v]}$.

Write $E' = v(F) \subset E^*$, $F' = v(E) \subset F^*$ and set $\ell_E = E^*/E'$, $\ell_F = F^*/F'$. Then, using v to identify $F' \simeq (E')^*$, one obtains $T_{[v]}\text{Zeros}(\det_n) = \ell_E \otimes F' \oplus (F')^* \otimes F' \oplus (F')^* \otimes \ell_F$ as a $\mathfrak{g}([v]) - module$. Write an element of $T_{[v]}\text{Zeros}(\det_n)$ as a triple (x, A, y). In matrices,

$$v = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad T_{[v]} \sim \begin{pmatrix} A & y \\ x & 0 \end{pmatrix}$$

Taking the $\mathfrak{g}([v])$-module structure into account, it is straight-forward to show:

Theorem 2.4.1. *Let* $X = \text{Zeros}(\det_n) \subset \mathbb{P}^{n^2-1} = \mathbb{P}(E \otimes F)$, *let* $v = e_1 \otimes f_1 + \cdots + e_{n-1} \otimes f_{n-1} \in X$. *With the notations above, the differential invariants of* X *at* $[v]$ *are*

$$\text{Zeros}(F_{2,[v]}(X)) = \text{Zeros}(xy)$$
$$\text{Zeros}(F_{2,[v]}(X), F_{3,x}(X)) = \text{Zeros}(xy, xAy)$$
$$\vdots$$
$$\text{Zeros}(F_{2,[v]}(X), ..., F_{k,x}(X)) = \text{Zeros}(xy, xAy, ..., xA^{k-2}y).$$

Since the permanent hypersurface is not quasi-homogeneous, its differential invariants are more difficult to calculate. It is even difficult to write down a general point in a nice way (that depends on m, keeping in mind that we are not concerned with individual hypersurfaces, but sequences of hypersurfaces). For example, the point on the permanent hypersurface chosen in [23] is not general as there is a finite group that preserves it. To get lower bounds it is sufficient to work with any point of the permanent hypersurface, but one will not know if the obtained bounds are sharp. To arrive at n being an exponential function of m, one might expect to improve the exponent by one at each order of differentiation. The following theorem shows that this does not happen at order three.

The Mignon-Ressayre result implies that any hypersurface in $2n - 2$ variables defined by a homogeneous polynomial can be approximated to order two at any point by an affine linear projection of $\{\det_n = 0\} \subset \mathbb{C}^{n^2}$.

Theorem 2.4.2. *Any hypersurface in* $n - 1$ *variables can be approximated to order three at any point by an affine linear projection of* $\{\det_n = 0\} \subset \mathbb{C}^{n^2}$.

In particular, $\{\text{perm}_m = 0\} \subset \mathbb{C}^{m^2}$ *can be approximated to order three at a general point by an affine linear projection of* $\{\det_{m^2+1} = 0\} \subset \mathbb{C}^{(m^2+1)^2}$.

Proof. The rank of F_2 for the determinant is $2(n-1)$, whereas the rank of F_2 for the permanent, and a deprojectivization of a general hypersurface in q variables at a general point is $q - 2$. so one would need to project to eliminate $(n - 1)^2$ variables to agree to order two.

Thus it is first necessary to perform a projection so that the matrix A, which formerly had independent variables as entries is now linear in the entries of x, y, write $A = A(x, y)$. Then the

projected pair F_2, F_3 is not generic because it has two linear spaces of dimension $n-1$ in its zero set. This can be fixed by setting $y = L(x)$ for $L : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ a linear isomorphism. At this point one has $F_2 = L(x)x$, $F_3 = L(x)A(x, L(x))x$. Take L to be the identity map, so the cubic is of the form $\sum_{i,j} x_i A_{ij}(x) x_j$ where the $A_{ij}(x)$ are arbitrary. This is an arbitrary cubic.   □

3. Detour: Grassmannians and Spinor varieties

3.1. **The Grassmannian as a variety parametrizing minors of matrices.** I use the summation convention that repeated indices appearing up and down are to be summed over their range.

Let W be a vector space and let $G(k, W)$ denote the Grassmannian of k-planes through the origin in W. Assume WLOG that $k \leq \dim W - k$. Recall the Plücker embedding $G(k, W) \subset \mathbb{P}(\Lambda^k W)$.

The cone over the Grassmannian, $\hat{G}(k, W) \subset \Lambda^k W$ is the set of $v \in \Lambda^k W$, such that there exist $w_1, ..., w_k \in W$ with $v = w_1 \wedge \cdots \wedge w_k$.

The Grassmannian $G(k, W)$ admits a local parametrization as follows: Write $W = E \oplus F$ where $\dim E = k$. Let $1 \leq i, j \leq k$, $1 \leq s, t \leq n - k$, fix bases $e_1, ..., e_k$ of E with dual basis $e^1, ..., e^k$ of E^*, and $f_1, ..., f_{n-k}$ of F with dual basis $f^1, ..., f^{n-k}$. Say $E = [v_0]$, $v_0 \in \hat{G}(k, W)$ and we want to locally parametrize $G(k, W)$ around $[v_0]$. Choose our basis such that $e_j = w_j$ in the description of v above. Let x_j^s be linear coordinates on $E^* \otimes F \simeq T_{[v_0]} G(k, W)$. The local parametrization about $E = [v(0)]$ is

$$[v(x_i^s)] = [(e_1 + x_1^s e_s) \wedge \cdots \wedge (e_k + x_k^s e_s)].$$

In what follows I will also need to work with $G(k, W^*)$, In our dual bases, a local parametrization about $E^* = \langle e^1, ..., e^k \rangle = [\alpha(0)]$ is

$$[\alpha(y_j^s)] = [(e^1 + y_s^1 e^s) \wedge \cdots \wedge (e^k + y_s^k e^s)].$$

I next explain how to interpret the open subset of $G(k, W)$ described above as the vector of minors for $E^* \otimes F$.

For vector spaces E, F, $\Lambda^k(E \oplus F)$ has the following decomposition as a $GL(E) \times GL(F)$ module:

$$\Lambda^k(E \oplus F) = (\Lambda^k E \otimes \Lambda^0 F) \oplus (\Lambda^{k-1} E \otimes \Lambda^1 F) \oplus (\Lambda^{k-2} E \otimes \Lambda^2 F)$$
$$\oplus \cdots \oplus (\Lambda^1 E \otimes \Lambda^{k-1} F) \oplus (\Lambda^0 E \otimes \Lambda^k F)$$

Assume we have a volume form on E so we may identify $\Lambda^s E \simeq \Lambda^{k-s} E^*$. We have the $SL(E) \times GL(F)$ decomposition:

$$\Lambda^k(E \oplus F) = (\Lambda^0 E^* \otimes \Lambda^0 F) \oplus (\Lambda^1 E^* \otimes \Lambda^1 F) \oplus (\Lambda^2 E^* \otimes \Lambda^2 F)$$
$$\oplus \cdots \oplus (\Lambda^{k-1} E^* \otimes \Lambda^{k-1} F) \oplus (\Lambda^k E^* \otimes \Lambda^k F)$$

Recall that $\Lambda^s E^* \otimes \Lambda^s F \subset S^s(E^* \otimes F)$ has the geometric interpretation as the space of $s \times s$ minors on $E \otimes F^*$, i.e., with any choices of bases, write an element f of $E \otimes F^*$ as a matrix, then a basis of $\Lambda^s E^* \otimes \Lambda^s F$ evaluated on f will give the set of $s \times s$ minors of f.

To see these minors explicitly, note that the bases of E^*, F induce bases of the exterior powers. Expanding out v above in such bases, (recall that the summation convention is in use)

$$v(x_i^s) = e_1 \wedge \cdots \wedge e_k$$
$$+ x_i^s e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \wedge e_k$$
$$+ (x_i^s x_j^t - x_j^s s_i^t) e_1 \wedge \cdots \wedge e_{i-1} \wedge e_s \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_t \wedge e_{j+1} \cdots \wedge e_k$$
$$+ \cdots$$

i.e., writing v as a row vector in the induced basis:

$$v = (1, x_i^s, x_i^s x_j^t - x_i^t x_j^s, \dots) = (1, \Delta_{i,s}(x), \dots, \Delta_{I,S}(x), \dots,)$$

where we use the notation $I = (i_1, \dots, i_p)$ $S = (s_1, \dots, s_p)$ and $\Delta_{I,S}(x)$ denotes the corresponding $p \times p$ minor of x. Similarly $\alpha = (1, y_s^j, y_s^j y_t^i - y_s^i y_t^j, \dots,)$.

Fix bases so x, y are $k \times (n-k)$ matrices. I claim

(3.1.1) $$\langle \alpha, v \rangle = \det(I_E + {}^t xy)$$

because the characteristic polynomial of a product of a $k \times \ell$ matrix ${}^t x$ with an $\ell \times k$ matrix y is:

(3.1.2) $$charpoly({}^t xy)(t) = \det(Id_E + t^t xy) = \sum_{I,S} \Delta_{I,S}(x) \Delta_{S,I}(y) t^{|I|}.$$

While (3.1.2) is no doubt classical, I include a proof as I didn't find one in the literature.

For a linear map $f : A \to A$, recall the induced linear maps $f^{\wedge k} : \Lambda^k A \to \Lambda^k A$, where, if one chooses a basis of A and represents f by a matrix, then the entries of the matrix representing $f^{\wedge k}$ in the induced basis on $\Lambda^k A$ will be the $k \times k$ minors of the matrix of f. In particular, if $\dim A = \mathbf{a}$, then, $f^{\wedge \mathbf{a}}$ is multiplication by a scalar which is $\det(f)$.

Recall the decomposition:

$$End(E^* \oplus F) = (E^* \otimes F) \oplus (E^* \otimes E) \oplus (F \otimes F^*) \oplus (F^* \otimes E).$$

To each $x \in E^* \otimes F$, $y \in E \otimes F^*$, associate the element

(3.1.3) $$-x + Id_E + Id_F + y \in End(E^* \oplus F).$$

Note that

$$\det \begin{pmatrix} I_E & -{}^t x \\ y & I_F \end{pmatrix} = \det(I_E + {}^t xy).$$

Consider

$$(-x + Id_E + Id_F + y)^{\wedge n} = (Id_E)^{\wedge k} \wedge (Id_F)^{\wedge(n-k)} + (Id_E)^{\wedge k-1} \wedge (Id_F)^{\wedge(n-k-1)} \wedge (-x) \wedge y$$

$$+ (Id_E)^{\wedge(k-2)} \wedge (Id_F)^{\wedge(n-k-2)} \wedge (-x)^{\wedge 2} \wedge y^{\wedge 2} + \cdots + (Id_F)^{\wedge(n-2k)} \wedge (-x)^{\wedge k} \wedge y^{\wedge k}$$

Let

$$e^1 \wedge \cdots \wedge e^k \wedge f_1 \wedge \cdots \wedge f_{n-k} \in \Lambda^n(E^* \otimes F)$$

be a volume form. All that remains to check is that when we re-order our terms that the signs work out correctly, which is left to the reader.

3.2. Spinor varieties.

For the interpretation of spinor varieties as maximal isotropic subspaces on a quadric, see any of [13, 15, 19]. Here I simply define the spinor variety as the Zariski closure of the set of vectors of sub-Pfaffians of a skew-symmetric matrix with variables as entries. See [18] for the connection with the classical definition.

For $x \in \Lambda^2 \mathbb{C}^{2n}$, the Pfaffian $\mathrm{Pf}(x) \in \mathbb{C}$ is defined by $x^{\wedge n} = \mathrm{Pf}(x) n! \Omega$, where $\Omega \in \Lambda^{2n} \mathbb{C}^{2n}$ is a volume form - it is a square root of $\det(x)$.

Let E be an n-dimensional vector space equipped with a volume form. Define $(\hat{\mathbb{S}}_+)^0$ to be the image of the map

$$\Lambda^2 E \to \Lambda^{even} E =: \mathcal{S}_+$$

$$x \mapsto v = (1, x_j^i, \dots, \mathrm{Pf}_I(x), \dots,) =: \mathrm{sPf}(x)$$

as $|I|$ varies over the even numbers from 0 to $\lfloor \frac{n}{2} \rfloor$. The space of sub-Pfaffians of size $2p$ is parametrized by $\Lambda^{2p} E$. If n is even, $\mathcal{S}_+$ is self dual, and if n is odd, its dual is $\mathcal{S}_- := \Lambda^{odd} E$ because E is equipped with a volume form, so $\Lambda^{2p} E^* = \Lambda^{n-2p} E$.

Recall the decomposition

$$\Lambda^2(E \oplus E^*) = \Lambda^2 E \oplus E \otimes E^* \oplus \Lambda^2 E^*.$$

Consider $x + Id_E + y \in \Lambda^2(E \oplus E^*)$. Observe that

$$(x + Id_E + y)^{\wedge n} = \sum_{j=0}^{n} (Id_E)^{\wedge(n-j)} \wedge x^{\wedge j} \wedge y^{\wedge j} \in \Lambda^{2n}(E \oplus E^*)$$

Let $\Omega = e_1 \wedge e^1 \wedge e_2 \wedge e^2 \wedge \cdots \wedge e_n \wedge e^n \in \Lambda^{2n}(E \oplus E^*)$ be a volume form. The coefficient of the j-th term is the sum

$$\sum_{|I|=2j} \operatorname{sgn}(I) \operatorname{Pf}_I(x) \operatorname{Pf}_I(y).$$

where for an even set $I \subseteq [n]$, define $\sigma(I) = \sum_{i \in I} i$, and define $\operatorname{sgn}(I) = (-1)^{\sigma(I)+|I|/2}$. Put more invariantly, the j-th term is the pairing

$$\langle y^{\wedge j}, x^{\wedge j} \rangle.$$

For a matrix z define a matrix $\tilde{z}$ by setting $\tilde{z}^i_j = (-1)^{i+j+1} z^i_j$. Let z be an $n \times n$ skew-symmetric matrix. Then for every even $I \subseteq [n]$,

$$\operatorname{Pf}_I(\tilde{z}) = \operatorname{sgn}(I) \operatorname{Pf}_I(z).$$

For $|I| = 2p$, $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$,

$$\operatorname{Pf}_I(\tilde{z}) = (-1)^{i_1+i_2+1} \cdots (-1)^{i_{2p-1}+i_{2p}+1} \operatorname{Pf}_I(z) = \operatorname{sgn}(I) \operatorname{Pf}_I(z).$$

Thus:

Theorem 3.2.1. [20] *Let z, y be skew-symmetric $n \times n$ matrices. Then*

$$\langle \operatorname{sPf}(z), \operatorname{sPf}^{\vee}(y) \rangle = \operatorname{Pf}(\tilde{z} + y).$$

In particular, the pairing $\mathcal{S}_+ \times \mathcal{S}_+^* \to \mathbb{C}$ restricted to $\hat{\mathbb{S}}_n \times \hat{\mathbb{S}}_{n*} \to \mathbb{C}$ can be computed in polynomial time.

The first few spinor varieties are classical varieties in disguise (corresponding to coincidences of Lie groups in the first two cases and triality in the third):

$$\mathbb{S}_2 = \mathbb{P}^2 \subset \mathbb{P}^2$$
$$\mathbb{S}_3 = \mathbb{P}^3 \subset \mathbb{P}^3$$
$$\mathbb{S}_4 = Q^6 \subset \mathbb{P}^7$$

In particular, although the codimension grows very quickly, it is small in these cases. The next case $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is not isomorphic to any classical homogeneous variety.

4. Holographic algorithms and spinors

4.1. **Counting problems as vector space pairings $A^* \times A \to \mathbb{C}$.** For simplicity of exposition, I restrict to the complexity problem of counting the number of solutions to equations c_s over $\mathbb{F}_2$ with variables x_i. This problem is called SAT in the complexity literature. (In complexity theory one usually deals with Boolean variables and clauses, which is essentially equivalent to equations over $\mathbb{F}_2$ but some care must be taken in the translation.)

To convert a counting problem to a vector space pairing, proceed as follows:

Step 1. To an instance of a problem construct a bipartite graph $\Gamma = (V_x, V_c, E)$ that encodes the problem. Here V_x, V_c are the two sets of vertices and E is the set of edges. V_x corresponds to the set of variables, V_c to the set of equations, and there is an edge e_{is} joining the vertex of the variable x_i to the vertex of the equation c_s iff x_i appears in c_s.

Step 2. Construct "local" tensors that encode the information at each vertex. To do this first associate to each edge e_{is} a vector space $A_{is} = \mathbb{C}^2$ with basis $a_{is|0}, a_{is|1}$ and dual basis $\alpha_{is|0}, \alpha_{is|1}$ of A_{is}^*. Next, to each variable x_i associate the vector space

$$A_i := \bigotimes_{\{s|e_{is}\in E\}} A_{is}$$

and the tensor

(4.1.1) $$g_i := \otimes_{\{s|e_{is}\in E\}} a_{is|0} + \otimes_{\{s|e_{is}\in E\}} a_{is|1} \in A_i$$

which will encode that x_i should be consistently assigned either 0 or 1 each time it appears. Now to each equation c_s we associate a tensor in $A_s^* := \otimes_{\{i|e_{is}\in E\}} A_{is}^*$ that encodes that c_s is satisfied. For example, say x_i, x_j, x_k appear in c_s and that

$$c_s(x_i, x_j, x_k) = x_i x_j + x_i x_k + x_j x_k + x_i + x_j + x_k + 1$$

which is satisfied over $\mathbb{F}_2$ as long as the variables x_i, x_j, x_k are not all 0 or all 1. (This equation is called *3NAE* in the computer science literature.) More generally, say c_s has $x_{i_1}, ..., x_{i_{d_s}}$ appearing and c_s is $d_s NAE$, then one associates the tensor

(4.1.2) $$r_s := \sum_{(\epsilon_1,...,\epsilon_{d_s}) \neq (0,...,0),(1,...,1)} \alpha_{i_1,s|\epsilon_1} \otimes \cdots \otimes \alpha_{i_{d_s},s|\epsilon_{d_s}}.$$

Step 3. Tensor all the local tensors from V_x (resp. V_c) together to get two tensors in dual vector spaces with the property that their pairing counts the number of solutions. That is, consider $G := \otimes_i g_i$ and $R := \otimes_s r_s$ respectively elements of the vector spaces $A := \otimes_e A_e$ and $A^* := \otimes_e A_e^*$. Then, the pairing $\langle G, R \rangle$ counts the number of solutions.

Remark 4.1.1. Up until now I could have just taken each $A_{is} = \mathbb{Z}_2$. The reason I used complex numbers was to allow a group action. This group action will destroy the local structure but leave the global structure unchanged. Valiant's inspiration for doing this was quantum mechanics, where particles are replaced by wave functions.

Also note that so far we have replaced our original counting problem with the problem of computing a pairing $A \times A^* \to \mathbb{C}$ where the dimension of A is exponential in the size of the input data. If we had arbitrary vectors, then there is no way to perform this pairing in a number of steps that is polynomial in the size of the original data. We saw that if one is lucky, the pairing can be computed quickly. We now try to make local changes of bases that simultaneously put each g_i and r_s into spinor varieties.

4.2. **Computing the vector space pairing in polynomial time.** To try to move both G, R to special position so that the pairing can be evaluated quickly, identify all the A_e with a single $\mathbb{C}^2$, and allow $SL_2\mathbb{C}$ to act. This action is very cheap, and of course if we have it act simultaneously on A and A^*, the pairing $\langle G, R \rangle$ will be unchanged. This step cannot always be carried out, otherwise we would have proved **P = NP**.

To illustrate, we now restrict to $3SAT - NAE$, which is still **NP**-hard.

The tensor g_i corresponding to a variable vertex x_i is (4.1.1). The tensor corresponding to a NAE clause r_s is (4.1.2) and $d_s = 3$ for all s. Let

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

be the basis change, the same in each A_e, sending $a_{e|0} \mapsto a_{e|0} + a_{e|1}$ and $a_{e|1} \mapsto a_{e|0} - a_{e|1}$ which induces the basis change $\alpha_{e|0} \mapsto \frac{1}{2}(\alpha_{e|0} + \alpha_{e|1})$ and $\alpha_{e|1} \mapsto \frac{1}{2}(\alpha_{e|0} - \alpha_{e|1})$ in A_e^*. Applying T,

gives

$$T(a_{i,s_{i_1}|0}\otimes\cdots\otimes a_{i,s_{i_{d_i}}|0}+a_{i,s_{i_1}|1}\otimes\cdots\otimes a_{i,s_{i_{d_i}}|1}) = 2\sum_{\{(\epsilon_1,...,\epsilon_{d_i})|\sum\epsilon_\ell=0\,(\mathrm{mod}\,2)\}} a_{i,s_{i_1}|\epsilon_1}\otimes\cdots\otimes a_{i,s_{i_{d_i}}|\epsilon_{d_i}}.$$

Similarly

$$T\left(\sum_{(\epsilon_1,\epsilon_2,\epsilon_3)\neq(0,0,0),(1,1,1)}\alpha_{i_1,s|\epsilon_1}\otimes\alpha_{i_2,s|\epsilon_2}\otimes\alpha_{i_3,s|\epsilon_3}\right)$$
$$= 6\alpha_{i_1,s|0}\otimes\alpha_{i_2,s|0}\otimes\alpha_{i_3,s|0} - 2(\alpha_{i_1,s|0}\otimes\alpha_{i_2,s|1}\otimes\alpha_{i_3,s|1} + \alpha_{i_1,s|1}\otimes\alpha_{i_2,s|0}\otimes\alpha_{i_3,s|1} + \alpha_{i_1,s|1}\otimes\alpha_{i_2,s|1}\otimes\alpha_{i_3,s|0})$$

After this change of basis $g_i \in \mathbb{S}_{\#\{s|e_{is}\in E\}}$ and $r_s \in \mathbb{S}_4$ for all i, s!

4.3. **NP, in fact #P is pre-holographic.**

Definition 4.3.1. Let P be a counting problem. We will say that P is *pre-holographic* if it admits a formulation such that the vectors g_i, r_s are all simultaneously representable as vectors of sub-Pfaffians.

The following was proved (although not stated) in [20]:

Theorem 4.3.2. *Any problem in* **NP**, *in fact* #**P** *is pre-holographic.*

Proof. To prove the theorem it suffices to exhibit one #**P** complete problem that is pre-holographic. Counting the number of solutions to $3SAT - NAE$ is one such. $\square$

4.4. **What goes wrong.**
While for $3SAT - NAE$ it is always possible to give V and V^* structures of the spin representations $\mathcal{S}_+$ and $\mathcal{S}_+^*$, so that $[G] \in \mathbb{P}V$ and $[R] \in \mathbb{P}V^*$ both lie in spinor varieties, these structures *may not be compatible*! What goes wrong is that the ordering of pairs of indices (i, s) that is good for V may not be good for V^*. The "only" thing that can go wrong are the signs of the sub-Pfaffians, see [20] for details. Since the same problem occurs in the case of the permanent, one might speculate that the difference between **P** or **VP** and **NP** or **VNP** is a matter of a finite group.

In [20] we determine sufficient conditions for there to be a good ordering of indices and show that if the bipartite graph Γ was planar, then these sufficient conditions hold.

4.5. **History.**
In Valiant's original formulation of holographic algorithms (see [35, 36, 37, 38, 39, 40]), the step of forming Γ is the same, but then Valiant replaced the vertices of Γ with weighted graph fragments to get a new weighted graph Γ' in such a way that the number of (weighted) perfect matchings of Γ' equals the answer to the counting problem. Then, if Γ' is planar, one can appeal to the famous FKT algorithm [17, 32] to compute the number of weighted perfect matchings in polynomial time. Valiant also found certain algebraic identities that were necessary conditions for the existence of such graph fragments.

Cai [6, 7, 8, 9, 10, 11, 12] recognized that Valiant's procedure could be reformulated as a pairing of tensors as in steps two and three, and that the condition on the vertices was that the local tensors g_i r_s could, possibly after a change of basis, be realized as a vector of sub-Pfaffians. In Cai's formulation one still appeals to the existence of Γ' and the FKT algorithm in the last step.

5. EXPONENTIAL PAIRINGS IN POLYNOMIAL TIME

5.1. **Cominscule varieties.**
In this section we show that the same phenomenon that we observed above for Grassmannians and spinor varieties holds for all cominuscule varieties.

Theorem 5.1.1. *Let V be a vector space of dimension $\binom{n}{k}$, 2^{n-1}, $\binom{2n}{k} - \binom{2n}{k-2}$, p^n, or $\binom{n+p-1}{n}$. In each case there are explicit systems of degree two polynomial equations on V, V^*, such that if $\alpha \in V^*$ and $v \in V$ satisfy these equations, the pairing $\langle \alpha, v \rangle$, which naïvely requires $O(\dim V)$ arithmetic operations, can be computed in $O(n^4)$ operations.*

Theorem 5.1.1 is an immediate consequence of:

Theorem 5.1.2. *Let $V = V(n)$ be a cominuscule $G = G(n)$-module with $G/P \subset \mathbb{P}V$ the closed orbit and $G/P' \subset \mathbb{P}V^*$ the corresponding closed orbit in the dual space. Here n is the rank of G. Then the pairing $V \times V^* \to \mathbb{C}$ restricted to $\hat{G}/P \times \hat{G}/P'$ can be computed in $O(n^4)$ arithmetic operations without divisions.*

The non-trivial cases are (where for notational convenience we use the rank of G plus one in the $A_{n-1} = SL_n$-case):

V	$\dim V$	G	G/P	$\mathfrak{g}/\mathfrak{p}$
$\Lambda^k W$	$\binom{n}{k}$	$SL(W) = SL_n$	$G(k, W)$	$E^* \otimes F$
$\mathcal{S}_+$	2^{n-1}	$D_n = Spin_{2n}$	$\mathbb{S}_+$	$\Lambda^2 E$
$\Lambda^{\langle n \rangle} W$	$\binom{2n}{n} - \binom{2n}{n-2}$	$Sp(2n, \mathbb{C}) = Sp(W, \omega)$	$G_{Lag}(n, 2n)$	$S^2 E$
$E_1 \otimes \cdots \otimes E_n$	p^n	$SL(E_1) \times \cdots \times SL(E_n)$	$Seg(\mathbb{P}E_1 \times \cdots \times \mathbb{P}E_n)$	$\oplus_j E'_j$
$S^n E$	$\binom{n+p-1}{n}$	$SL(E)$	$v_n(\mathbb{P}E)$	$E' \circ \ell^{n-1}$

Explanations of V: W is a vector space of dimension n in the first case, $2n$ in the third, $\mathcal{S}_+$ is the (positive) half-spin representation of $Spin_{2n}$, $\Lambda^{\langle n \rangle} W = \Lambda^n W / (\Lambda^{n-2} W \wedge \omega)$ where $\omega \in \Lambda^2 W$ is a symplectic form. E, E_j are vector spaces of dimension p in the last two cases.

Explanations of G/P: $G(k, W)$ denotes the Grassmannian of k-planes in its Plücker embedding, $\mathbb{S}_+$ the "pure spinors" or *spinor variety*, $G_{Lag}(n, 2n)$ denotes the Lagrangian Grassmannian of n-planes isotropic for the symplectic form $\omega \in \Lambda^2 \mathbb{C}^{2n}$, $Seg(\mathbb{P}E_1 \times \cdots \times \mathbb{P}E_n)$ denotes the Segre product, the projectivization of the set of decomposable tensors in $E_1 \otimes \cdots \otimes E_n$ and $v_n(\mathbb{P}E)$ denotes the Veronese variety of the projectivization of homogeneous polynomials of degree n on E^* that are n-th powers of a linear form.

Explanations of $\mathfrak{g}/\mathfrak{p}$: $\mathfrak{g}, \mathfrak{p}$ are the Lie algebras of G, P. Let G_0 denote the Levi-factor of P. G_0 is respectively $S(GL(E) \times GL(F))$, $GL(E)$, $GL(E)$, $GL(E'_1) \times \cdots \times GL(E'_n)$, $GL(E')$. As a G_0-module, $\mathfrak{g}/\mathfrak{p}$ is the tangent space to G/P at the point of G/P corresponding to $Id \in G$. I have written $F = W/E$. Fix vectors $e_j \in E_j$, $e \in E$ and let ℓ_j, ℓ respectively denote the lines they span, then $E'_j = \ell_1 \otimes \cdots \otimes \ell_{j-1} \otimes E_j / \ell_j \otimes \ell_{j+1} \otimes \cdots \otimes \ell_n$ and $E' = E/\ell$.

In each case $\mathfrak{g}/\mathfrak{p}$ is a space of endomorphisms and V as a G_0-module is the sum of the spaces of all minors (of all sizes) of $\mathfrak{g}/\mathfrak{p}$, except in the spinor case, where one takes all sub-Pfaffians. $\oplus_j S_{2j} E = \oplus_j S_{2\ldots 2} E$ denotes the irreducible $GL(E)$-submodule of $\Lambda^j E \otimes \Lambda^j E$ giving minors on $S^2 E \subset E \otimes E$.

It remains to prove the cases of the Lagrangian Grassmannian, the Segre and the Veronese.

5.2. Lagrangian Grassmannian case.

The Lagrangian Grassmannian $G_{Lag}(n, 2n) \in \mathbb{P}\Lambda^{\langle n \rangle} W$ is a linear section of $G(n, 2n) \subset \mathbb{P}\Lambda^n W$. Here $\Lambda^{\langle n \rangle} W = \Lambda^n W / (\Lambda^{n-2} W \wedge \omega) = W_{\omega_n}^{Sp(2n, W)}$ and the quotient may be viewed as the complement to $\Lambda^{n-2} W \wedge \omega \subset \Lambda^n W$ to obtain the linear section.

The interpretation of an open subset (the "big cell") of $G_{Lag}(n, W)$ is as the set of vectors of (non-redundant) minors of symmetric matrices. The symplectic form enables the identification of $W/E \simeq E^*$ and the linear subspace of

$$E^* \otimes E^* = \Lambda^2 E^* \oplus S^2 E^*$$

corresponding to the tangent space is just $S^2 E^*$. See [19] for details.

The subspace of $\Lambda^j E^* \otimes \Lambda^j E^*$ giving rise to a non-redundant set of minors corresponds to the sub-module $S_{2^j} E^* \subset \Lambda^j E^* \otimes \Lambda^j E^*$.

For the Lagrangian Grassmannian case it suffices in (3.1.3) to take

$$-x + Id_E + y \in S^2(E \oplus E^*) = S^2 E \oplus E \otimes E^* \oplus S^2 E^*.$$

5.3. Segre and Veronese cases.

The Segre is parametrized by a map ϕ

$$(x_s^j) \mapsto (a_0^1 + x_1^j a_j^1) \otimes (a_0^2 + x_2^j a_j^2) \otimes \cdots \otimes (a_0^n + x_n^j a_j^n) = (1, x_{s_1}^j, x_s^j x_{s_2}^k, \cdots x_{s_1}^1 \cdots x_{s_p}^p),$$

where in each term $s_1 < \cdots < s_q$. Let $\phi^\vee$ denote the map to the dual Segre.

If $\alpha = \phi(x)$, $v = \phi^\vee(y)$ then

$$\langle \alpha, v \rangle = \sum_{I,S} x_S^I y_S^I$$

where $I = (i_1, ..., i_q)$, $i_1 \leq \cdots \leq i_q$, $1 \leq q \leq p$, and $S = (s_1, ..., s_r)$, $s_1 < \cdots < s_r$, $1 \leq r \leq n$. Here :

$$\langle \alpha, v \rangle = \det \begin{pmatrix} & -x_1^1 & & -x_1^2 & & & -x_1^p & \\ I_n & & \ddots & & \ddots & \cdots & & \ddots \\ & -x_n^1 & & -x_n^2 & & & -x_n^p & \\ y_1^1 & & & & & & & \\ \ddots & & I_n & & & & & \\ y_n^1 & & & & & & & \\ y_1^2 & & & & & & & \\ \ddots & & & & I_n & & & \\ y_n^2 & & & & & & & \\ \vdots & & & & & & \ddots & \\ y_1^p & & & & & & & \\ \ddots & & & & & & & I_n \\ y_n^p & & & & & & & \end{pmatrix}.$$

The Veronese is parametrized by $(x^j) \mapsto (a_0 + x^j a_j)^p$ and the same matrix as above works replacing x_s^j with x^j for all s and similarly for y.

6. Definitions of VP, VNP and VP$_e$

6.1. VP$_e$.

An elementary measure of the complexity of a (homogeneous) polynomial p is as follows: given an expression for p, count the total number of additions plus multiplications present in the expression, and then take the minimum over all possible expressions.

Aside 6.1.1. Here there are two types of multiplications, multiplying a variable by a variable, and a variable by a scalar. For questions we will be concerned with it doesn't matter whether or not one counts the multiplications of a scalar times a variable, although, it is the multiplication of variables that is far more costly for a computer. Multiplication of variables is also more complicated than addition of variables - but the order of complexity will not be significant for us so we will count them both equally or just count multiplications.

Example 6.1.2.

$$p_n(x, y) = x^n + nx^{n-1}y + \binom{n}{2} x^{n-2} y^2 + \binom{n}{3} x^{n-3} y^3 + \cdots + y^n$$

This expression for p_n involves $n(n+1)$ multiplications and n additions, but one can also write

$$p_n(x, y) = (x + y)^n$$

which requires n multiplications and one addition to evaluate.

Definition 6.1.3. An *arithmetic circuit* C is a finite, acyclic, directed graph with veritices of in-degree 0 or 2 and exactly one vertex of out degree 0. In degree 0, inputs are labelled by elements of $\mathbb{C} \cup \{x_1, ..., x_n\}$ and in degree 2, vertices are called *computation gates* and labelled with $+$ or $*$. The *size* of C is the number of vertices. From a circuit C, one can construct a polynomial p_C in the variables $x_1, ..., x_n$.

If C is a tree (i.e., all out degrees are at most one), then the size of C equals the number of $+$'s and $*$'s used in the formula constructed from C.

Definition 6.1.4. For $f \in S^d\mathbb{C}^m$, the *expression size* $E(f)$ is the smallest size of a tree circuit that computes f. Define the class $\mathbf{VP}_e$ to be the set of sequences (p_n) such that there exists a sequence (C_n) of tree circuits, with the size of C_n bounded by a polynomial in n, such that C_n computes p_n.

It turns out that expression size is too naïve a measurement of complexity, as consider Example 6.1.2, we could first compute $z = x + y$, then $w = z^2$, then w^2 etc... until the exponent is close to n, for a significant savings in computation.

6.2. **VP, VP$_{ws}$ and closures.** Allowing circuits more general than trees allows us the possibility of remembering the results of previous calculation and gives rise to the class **VP**:

Definition 6.2.1. The class $\mathbf{VP}$ is the set of sequences (p_n) of polynomials of degree $d(n)$ in $v(n)$ variables where $d(n), v(n)$ are bounded by polynomials in n and such that there exists a sequence of circuits (C_n) of polynomially bounded size such that C_n computes p_n.

Definition 6.2.2. A problem P is *hard* for a complexity class $\mathbf{C}$ if all problems in $\mathbf{C}$ can be reduced to P (i.e. there is an algorithm to translate any instance of a problem to an instance of P with comparable input size). A problem P is *complete* for $\mathbf{C}$ if it is hard for $\mathbf{C}$ and if $P \in \mathbf{C}$.

A famous example of a sequence in $\mathbf{VP}$ is $\det_n \in S^n\mathbb{C}^{n^2}$, despite its apparently huge expression size. While it is known that $(\det_n) \in \mathbf{VP}$, it is not known whether or not it is $\mathbf{VP}$-complete. On the other hand, it is known that $(\det_n)$ is $\mathbf{VP}_e$-hard, although it is not known whether or not $(\det_n) \in \mathbf{VP}_e$. When complexity theorists and mathematicians are confronted with such a situation, what else do they do other than make another definition?

Definition 6.2.3. The class $\mathbf{VP}_{ws}$ is the set of sequences (p_n) where $\deg(p_n)$ is bounded by a polynomial and such that there exists a sequence of circuits (C_n) of polynomially bounded size such that C_n represents p_n, and such that at any multiplication vertex, the component of the circuit of one of the two edges coming in is disconnected from the rest of the circuit by removing the multiplication vertex.

In [22] they show that $(\det_n)$ is $\mathbf{VP}_{ws}$-complete, so Conjecture 2.3.3 may be rephrased as conjecturing $\mathbf{VP}_{ws} \neq \mathbf{VNP}$.

Remark 6.2.4. It is considered a major open question to determine whether or not $(\det_n) \in \mathbf{VP}_e$.

Definition 6.2.5. Given a complexity class $\mathbf{C}$ defined in terms of sequences of polynomials, we define a sequence (p_n) to be in $\overline{\mathbf{C}}$ if there exists a curve of sequences $q_{n,t}$, such that for each fixed $t_0 \neq 0$, $(q_{n,t_0}) \in \mathbf{C}$ and for all n, $\lim_{t \to 0} q_{n,t} = p_n$.

6.3. **VNP.** The class **VNP** essentially consists of polynomials whose coefficients can be determined in polynomial time. Consider a sequence $h = (h_n) \in \mathbb{C}[x_1, ..., x_n]_{\leq n}$ of (not necessarily homogeneous) polynomials of the form

$$(6.3.1) \qquad\qquad h_n = \sum_{e \in \{0,1\}^n} g_n(e) x_1^{e_1} \cdots x_n^{e_n}$$

where $(g_n) \in$ **VP**. Define **VNP** to be the set of all sequences that are projections of sequences of the form h. For equivalent definitions, see e.g., [4, §21.2].

Proposition 6.3.1. [33] $(\mathrm{perm}_n) \in$ **VNP**, *in fact is* **VNP**-*complete.*

Conjecture 6.3.2. [34] *[Valiant's hypothesis]* **VP** $\neq$ **VNP**.

   It is known that **P** $\neq$ **NP** would imply **VP** $\neq$ **VNP**.

7. GEOMETRIC COMPLEXITY THEORY APPROACH TO $\overline{\mathbf{VP}_{ws}}$ V. **VNP**

7.1. **Description of the program.** In a series of papers [30, 31, 28, 29, 27, 25, 26, 24], K. Mulmuley and M. Sohoni outline an approach to prove $\overline{\mathbf{VP}_{ws}} \neq$ **VNP**. Let ℓ be a linear coordinate on $\mathbb{C}$, and take any linear inclusion $\mathbb{C} \oplus \mathbb{C}^{m^2} \subset \mathbb{C}^{n^2}$ to have $\ell^{n-m}\mathrm{perm}_m$ be a homogeneous degree n polynomial on $\mathbb{C}^{n^2}$. Mulmuley and Sohoni observe that $\overline{\mathbf{VP}_{ws}} \neq$ **VNP** is equivalent to the assertion that the smallest function $\tilde{n}(m)$ such that $[\ell^{\tilde{n}-m}\mathrm{perm}_m] \in \overline{GL_{\tilde{n}^2} \cdot [\det_{\tilde{n}}]}$ grows faster than any polynomial, or equivalently:

Conjecture 7.1.1. [30] *Let $\tilde{n}(m)$ be the smallest number such that* $\overline{GL_{\tilde{n}^2} \cdot [\ell^{\tilde{n}-m}\mathrm{perm}_m]} \subset \overline{GL_{\tilde{n}^2} \cdot [\det_{\tilde{n}}]}$, *then $\tilde{n}(m)$ is not a polynomial in m.*

   For a complex projective variety $X \subset \mathbb{P}V$, let $I(X) \subset Sym(V^*)$ be the ideal of polynomials vanishing on X. Let $\mathbb{C}[X] = Sym(V^*)/I(X)$ denote the homogeneous coordinate ring. For complex projective varieties $X, Y \subset \mathbb{P}^N = \mathbb{P}V$, one has $X \subset Y$ iff $\mathbb{C}[Y]$ surjects onto $\mathbb{C}[X]$ (by restriction of functions). Mulmuley and Sohoni set out to prove:

Conjecture 7.1.2. [30] *Let $u(m)$ be a polynomial. There is a sequence of irreducible modules M_m for $GL_{u(m)^2}$ such that M_m appears in $\mathbb{C}[\overline{GL_{u(m)^2} \cdot [\ell^{u(m)-m}\mathrm{perm}_m]}]$ but not in $\mathbb{C}[\overline{GL_{u(m)^2} \cdot [\det_{u(m)}]}]$.*

   There are several paths one could take to try to find such a sequence of modules. The path Mulmuley and Sohoni choose in [31] is to consider $SL_{n^2} \cdot \det_n$ and $SL_{m^2} \cdot \mathrm{perm}_m$ because on the one hand their coordinate rings can be determined in principle using representation theory, and on the other hand they are closed affine varieties. They observe that any SL_{n^2}-module appearing in $\mathbb{C}[SL_{n^2} \cdot \det_n]$ must also appear in $\mathbb{C}[\overline{GL_{n^2} \cdot \det_n}]_k$ for some k. Regarding the permanent, for $n > m$, $SL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m$ is not closed, so they develop machinery to transport information about $\mathbb{C}[SL_{m^2} \cdot \mathrm{perm}_m]$ to $\mathbb{C}[\overline{GL_{n^2} \cdot \ell^{n-m}\mathrm{perm}_m}]$, including a notion of *partial stability*.

   Mathematical aspects of this program are discussed in [2]. The representation-theoretic information Mumuley and Sohoni propose to exploit is studied in detail. In particular [2, Thm 5.7.1] is a precise description of conditions on *Kronecker coefficients* that would imply Conjecture 7.1.2. In addition, suggestions are made for further geometric information that one could take into account that might imply a more tractible problem in representation theory.

   The price of using SL_{n^2} instead of GL_{n^2} is that one loses the grading of the coordinate rings. On the other hand, in order to use GL_{n^2}, one must solve, or at least partially solve, an *extension problem*, which to even begin work on, means that one must determine the codimension one components of the boundaries in the orbit closures. Nevertheless, it seems this harder work is essential as is explained in the following examples.

7.2. **Beyond determinant and permanent.** Instead of considering $\det_n$, one could take a sufficiently generic $g_n \in GL_{n^2}$ and consider $p_n := \det_n + g_n \cdot \det_n$. Then $G(p_n)$ will be the same as that for a generic polynomial, although p_n is still $\mathbf{VP}_{ws}$-complete. Thus just looking at the orbit, there would be *fewer* modules appearing in $\mathbb{C}[SL_{n^2} \cdot [p_n]]$ than in $\mathbb{C}[SL_{n^2} \cdot [\text{perm}_n]]$. In particular the orbit closure is larger than that of the permanent. More generally, let $r(n)$ be a polynomial and take a sequence of points in $p_n \in \sigma_{r(n)}(\overline{GL_{n^2} \cdot [\det_n]})$, the r-th secant variety of $\overline{GL_{n^2} \cdot [\det_n]}$. One could study the differential invariants of these varieties to see how they project to the permanent as in §2 and consider GCT program using the varieties $\overline{GL_{n^2} \cdot p_n}$.

More examples of sequences of polynomials are given by the *immanants* defined by Littlewood in [21]. Immanants generalize the determinant and permanent. Given a partition $\pi = (p_1, ..., p_r)$ of n, and a vector space V of dimension at least r, let $S_\pi V$ denote the corresponding irreducible $GL(V)$-module. $IM_\pi \in S^n \mathbb{C}^{n^2}$ may be defined as follows: consider $\mathbb{C}^{n^2} = E \otimes F$, where $E, F = \mathbb{C}^n$. Then $S^n(E \otimes F) = \oplus_\pi S_\pi E \otimes S_\pi F$ as a $GL(E) \times GL(F)$ module. Let $D^E \subset GL(E)$, $D^F \subset GL(F)$ denote the tori, i.e., the groups of diagonal matrices. Let $\mathfrak{S}_n^E, \mathfrak{S}_n^F$ denote the groups of permutation matrices acting on the left and right, and let $\Delta(\mathfrak{S}_n) \subset \mathfrak{S}_n^E \times \mathfrak{S}_n^F$ denote the diagonal embedding. Then $IM_\pi \in S_\pi E \otimes S_\pi F$ is the unique (up to scale) element acted on trivially by $(D^E \times D^F) \ltimes \Delta(\mathfrak{S}_n)$.

In [45], building on work in [14], it is shown that for all $\pi \neq (1^n), (n)$, that $G(IM_\pi) = ((D^E \times D^F) \ltimes \Delta(\mathfrak{S}_n)) \ltimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by sending a matrix to its transpose.

Consider $IM_{(n-1,1)}$ and $IM_{(2,1^{n-1})}$. The first is $\mathbf{VNP}$-complete and the second is in $\mathbf{VP}$, see [3], so we could attempt to apply the GCT program to them. By [45] $G(IM_{(n-1,1)}) = G(IM_{(2,1^{n-1})})$ so $\mathbb{C}[SL_{n^2} \cdot [IM_{(n-1,1)}]] = \mathbb{C}[SL_{n^2} \cdot [IM_{(2,1^{n-1})}]]$. Without examining the boundaries of $\overline{GL_{n^2} \cdot [IM_{(n-1,1)}]}$ and $\overline{GL_{n^2} \cdot [IM_{(2,1^{n-1})}]}$ there is no way to distinguish them.

Such investigations will be the subject of future work.

8. Towards geometric definitions of complexity classes

As mentioned several times, symmetry, sometimes in hidden form, appears to play a central role in characterizing sequences in $\mathbf{VP}$ that are apparently not in $\mathbf{VP}_e$. Another issue mentioned several times is that to make a good geometric study of complexity, one should have coordinate free definitions. In this section I give a coordinate free and geometric definition of the class $\overline{\mathbf{VP}}_e$. I then give a coordinate free and geometric definition of a class $\mathbf{VP}_{hs}$ which is perhaps a close approximation to $\mathbf{VP}$. Unfortunately at this writing I have no idea for a proposed purely geometric definition of $\mathbf{VNP}$. (S. Basu and M. Shub, in separate personal communications, have proposed that $\mathbf{VNP}$ should somehow be viewed as a bundle over $\mathbf{VP}$, but I have been unable to make this precise.)

8.1. **Joins and multiplicative joins.** The *join* of projective varieties $X_1, ..., X_r \subset \mathbb{P}V$, $J(X_1, ..., X_r) \subset \mathbb{P}V$, is the Zariski closure of the points of the form $[p_1 + \cdots + p_r]$ with $[p_j] \in X_j$. The expected dimension of $J(X_1, ..., X_r)$ is $\min(\sum \dim X_j + r - 1, \dim \mathbb{P}V)$. Let $\hat{T}_{[p]}X \subset V$ denote the affine tangent space of X at $[p] \in X$. Terracini's lemma says that if $([p_1], ..., [p_r]) \in X_1 \times \cdots \times X_r$ is a general point, then

$$\hat{T}_{[p_1 + \cdots + p_r]} J(X_1, ..., X_r) = \hat{T}_{[p_1]}X_1 + \cdots + \hat{T}_{[p_r]}X_r.$$

One can similarly define joins in affine space. The expressions are the same without the brackets.

Definition 8.1.1. Let $X \subset \mathbb{P}S^a V$, $Y \subset \mathbb{P}S^b V$ be varieties. Define the *multiplicative join* of X and Y, $MJ(X, Y)$, by

$$MJ(X, Y) := \{[p \circ q] \mid [p] \in X, \ [q] \in Y\} \subset \mathbb{P}S^{a+b}V.$$

For varieties $X_j \subset \mathbb{P}S^{d_j}V$, define $MJ(X_1, ..., X_r) \subset S^dV$ similarly (or inductively as $MJ(X, Y, Z) = MJ(X, MJ(Y, Z))$). In the special case $X_j = \mathbb{P}V \subset \mathbb{P}S^1V$, $MJ(\mathbb{P}V, ..., \mathbb{P}V)$ is the Chow variety of polynomials that decompose into a product of linear factors.

Similarly, let $A_{d,v}$ denote the space of all polynomials of degree at most d in v variables. For affine varieties $X \subset A_{d_1,v}$, $Y \subset A_{d_2,v}$, $MJ(X, Y) \subset A_{d_1+d_2,v}$ is defined in the same way without brackets.

Proposition 8.1.2. *Let $X_j \subset \mathbb{P}S^{d_j}V$ be varieties and let $([p_1], ..., [p_r]) \in X_1 \times \cdots \times X_r$ be a general point. Then*

$$\hat{T}_{[p_1 \circ \cdots \circ p_r]} MJ(X_1, ..., X_r) = \hat{T}_{[p_1]}X_1 \circ p_2 \circ \cdots \circ p_r + \cdots + p_1 \circ \cdots \circ p_{r-1} \circ \hat{T}_{[p_r]}X_r$$

In particular, the expected dimension of $MJ(X_1, ..., X_r)$ is $\min(\dim X_1 + \cdots + \dim X_r, \dim \mathbb{P}S^dV)$.

Proof. Let $p_j(t)$ be a curve in X_j with $p_j(0) = p_j$. Differentiate the expression $p_1(t) \circ \cdots \circ p_r(t)$ at $t = 0$ to get the result. $\qquad\square$

Question 8.1.3. What are the degenerate multiplicative joins, i.e., those that fail to be of the expected dimension?

8.2. A geometric characterization of $\overline{\textbf{VP}}_e$. Recall that the expression size $E(p)$ of a polynomial $p \in A_{d,v}$ is given by the number of internal nodes of the smallest tree circuit computing p. Define $\overline{E}(p)$ to be the smallest integer such that there is a curve p_t with $\lim_{t \to 0} p_t = p_0$ and such that $E(p_t) = \overline{E}(p)$ for $t \neq 0$. By definition, a sequence $(p_n) \in A_{d(n),v(n)}$ is in $\textbf{VP}_e$ (resp. $\overline{\textbf{VP}}_e$) if there exists a polynomial $r(n)$ such that $E(p_n) \leq r(n)$ (resp. $\overline{E}(p_n) \leq r(n)$.).

To a tree circuit Γ associated to a polynomial p associate an algebraic variety as follows: first form an new tree circuit Γ' by collapsing all pairs of input nodes that are joined by a $+$ to a single input node, and then repeating as many times as necessary until no pairs of input nodes are joined by a $+$. (I take this first step to eliminate the choice of coordinates involved in making the circuit.) Associate to each input node a copy of V.

Thus on Γ', if any two input nodes are joined, they are joined by a $*$-node. Now perform a step by step procedure to eliminate all $*$-nodes joining pairs of input nodes. Take a $*$-node joined to two input nodes, and form a subtree containing all other $*$-nodes joined to it and an input node. Say there are $j_1 - 1$ such. Record the variety $MJ_{j_1} := MJ(V, ..., V)$ of j_1 copies of V. Collapse the subtree to a single input node and associate MJ_{j_1} to this input node. Now start again, say we arrive at $j_2 - 1$ nodes in the subtree and record the variety $MJ_{j_2} = MJ(V, ..., V)$ of j_2 copies of V. Continue until we have recorded p varieties of multiplicative joins of V of various sizes.

We arrive at a new graph Γ'' all of whose p input nodes have varieties MJ_{j_i} associated to them and when input nodes are paired together by an internal node, the node is a $+$-node. Now perform a step by step procedure to eliminate all $+$'s joining pairs of input nodes. Take the first $+$, say that the variety $MJ_{j_{i_1}}$ is one of the input nodes and form a subtree consisting of all other $+$'s joined to it. Say there are $k - 1$ such. Record the variety $J(MJ_{j_{i_1}}, ..., MJ_{j_{i_k}})$. Collapse the subtree to a single input node and associate $J(MJ_{j_{i_1}}, ..., MJ_{j_{i_k}})$ to this input node. Continue until we have varieties of joins of multiplicative joins of various sizes as our new input nodes with all pairings of input nodes $*$-nodes.

Now continue as we did with Γ', taking multiplicative joins (of the joins of multiplicative joins) until the further collapsed graph has all pairings of input nodes $+$'s, then go back to taking joins etc...

This process terminates after a number of steps fewer than the number of nodes of Γ, and one arrives at a variety Σ_Γ of successive joins and multiplicative joins. By construction $p \in \Sigma_\Gamma$.

Note that for each such variety, there are many Γ that are associated to it, but each has, up to the initial v times the number of initial input nodes, the same expression size.

Let $\Sigma_R^{d,v}$ denote the union of all the varieties obtainable from a graph of at most R internal nodes computing an element of $A_{d,v}$. There is a finite number of such, so $\Sigma_R^{d,v}$ is an algebraic variety. The above discussion implies

Theorem 8.2.1. *Let $p_n \in A_{d(n),v(n)}$ be a sequence with d, v polynomials. Then $(p_n) \in \overline{\mathbf{VP}}_e$ iff there exists a polynomial $R(n)$ and $p_n \in \Sigma_{R(n)}^{d(n),v(n)}$. In other words the complexity class $\overline{\mathbf{VP}}_e$ is characterized by a sequence of algebraic varieties.*

Remark 8.2.2. One has to use the class $\overline{\mathbf{VP}}_e$ instead of $\mathbf{VP}_e$ because when taking joins one must include limits. It is not necessary to include limits when taking multiplicative joins.

Corollary 8.2.3. *A sequence $(p_n) \in A_{d(n),v(n)}$ is in $\overline{\mathbf{VP}}_e$ if either d or v is constant. A generic sequence in $A_{d(n),v(n)}$ is not in $\overline{\mathbf{VP}}_e$ if both d, v grow at least linearly with respect to n.*

Proof. $\dim \Sigma_R^{d,v} \leq (v+1)(R+1)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

8.3. **Towards a geometric understanding of VP.** It is not unreasonable to think that any (putative) difference between $\mathbf{VP}$ and $\mathbf{VP}_e$ is captured by the determinant, which has the property that for each n there is a subspace $\mathfrak{b}_n \subset \mathbb{C}^{n^2}$, such that $\det_n |_{\mathfrak{b}_n} \in \mathbf{VP}_e$ and moreover $G(\det_n) \cdot \mathfrak{b}_n = \mathbb{C}^{n^2}$. This perspective motivates the following definitions.

Define $\mathbf{VP}^{prim}$ to be the set of sequences $p_n \in A_{d(n),v(n)}$, where for each n, there exists a linear subspace $\Sigma_n \subset \mathbb{C}^{v(n)}$, such that the sequence $(p_n)|_{\Sigma_n}$ lies in $\mathbf{VP}_e$, and letting $G(n)$ denote the subgroup of $GL_{v(n)}$ preserving (p_n), ask moreover that $G(n) \cdot \Sigma_n = \mathbb{C}^{v(n)}$. Clearly $\mathbf{VP}^{prim} \subset \mathbf{VP}$ as the action of $G(n)$ is cheap. $\mathbf{VP}^{prim}$ is modeled on $(\det_n)$ where Σ_n is the upper-triangular matrices. Define $\mathbf{VP}_{hs}$ to be set of sequences (p_n) such that there exists another sequence (r_n) with $(r_n) \in \mathbf{VP}_e$, a polynomial $q(n)$, and sequences $(p_{n,j})$, $j = 1, ..., q(n)$ such that $(p_{n,j}) \in \mathbf{VP}^{prim}$ and $p(n) = r_n(p_{n,1}, ..., p_{n,q(n)})$. Then $\mathbf{VP}_{hs} \subseteq \mathbf{VP}$.

Question 8.3.1. What is the gap, if any, between $\mathbf{VP}$ and $\mathbf{VP}_{hs}$?

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