Chapter 1: The Measurement of Interest

Interest may be defined as the compensation that a borrower of capital pays to a lender of capital for its use.

Section 1.2: The Accumulation and Amount Functions

Principal is the initial amount of money invested.

Accumulated amount The total amount of money received after a period of time.

The amount of interest earned is the difference between the accumulated amount and the principal.

The period is the unit in which time is measured. The more common measurement period is one year, and will be assumed unless otherwise stated.

The accumulation function, denoted \(a(t)\), gives the accumulated value at time \(t \geq 0\) of an original investment of 1. The properties of \(a(t)\) are:

- \(a(0) = 1\)
- For positive interest rates, \(a(t)\) is an increasing function. Negative interest rates gives a decreasing function.
- If interest accrues continuously, the function will be continuous. Otherwise, \(a(t)\) will have discontinuities.

The amount function, denoted \(A(t)\), gives the accumulated value at time \(t \geq 0\) of an original investment of \(k\). The properties of \(A(t)\) are:

- \(A(t) = k \cdot a(t)\) so we get \(A(0) = k\)
- For positive interest rates, \(A(t)\) is an increasing function. Negative interest rates gives a decreasing function.
- If interest accrues continuously, the function will be continuous. Otherwise, \(A(t)\) will have discontinuities.
Define $I_n$ as the amount of interest earned in the $n$-th period from the date of investment.

$$I_n = A(n) - A(n - 1)$$ for $n = 1, 2, 3, \cdots$

**Proportionality:** Suppose an investment of $b$ is made and this investment will follow another investment strategy whose amount function is given by $A(t)$, $t > 0$.

- If the investment of $b$ is made at time $0$, then the value of the investment at time $t$ is given by $b \cdot \frac{A(t)}{A(0)}$.

- If the investment of $b$ is made at time $s$, then the value of the investment at time $t$ is given by $b \cdot \frac{A(t)}{A(s)}$.

Example: An investment of $10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A(t)$</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10,600</td>
<td>$I_1 = 600$</td>
</tr>
<tr>
<td>2</td>
<td>11,024</td>
<td>$I_2 = 424$</td>
</tr>
<tr>
<td>3</td>
<td>11,575.2</td>
<td>$I_3 = 551.2$</td>
</tr>
<tr>
<td>4</td>
<td>12,732.72</td>
<td>$I_4 = 1,157.52$</td>
</tr>
</tbody>
</table>

If $5,000 is invested at $t = 1$ under the same interest environment, find the accumulated value of the $5,000 at time $t = 3$. 
Example: It is known that $a(t)$ is of the form $a(t) = be^{0.1t} + c$. If $300$ invested at time $0$ accumulates to $309.73$ at time $5$, find the accumulated value at time $12$ of $250$ invested at time $3$.

Section 1.3: The Effective Rate of Interest

The effective rate of interest, $i$, is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period. This assumes the principal remains constant throughout the period.

Note: The effective rate of interest is often expressed as a percentage. Thus $7\% \equiv 0.07$ earned per unit of principal.

The effective rate of interest for a period can be defined in terms of the amount function by the following.

\[ i = \]
The effective rate of interest can be computed during the $n$-th period.

\[ i_n = \]

Example: An investment of $10,000 is made into a fund at time $t = 0$. The fund develops the following balances over the next 4 years. Find the effective rate of interest for each of the 3 years.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A(t)$</th>
<th>$I_n$</th>
<th>$i_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10,600</td>
<td>$I_1 = 600$</td>
<td>$i_1 =$</td>
</tr>
<tr>
<td>2</td>
<td>11,024</td>
<td>$I_2 = 424$</td>
<td>$i_2 =$</td>
</tr>
<tr>
<td>3</td>
<td>11,575.2</td>
<td>$I_3 = 551.2$</td>
<td>$i_3 =$</td>
</tr>
</tbody>
</table>
Section 1.4: Simple Interest

Simple interest is when the amount of interest earned during each period is constant.

<table>
<thead>
<tr>
<th>account value</th>
<th>1</th>
<th>1 + i</th>
<th>1 + 2i</th>
<th>1 + 3i</th>
<th>1 + 4i</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Thus \( a(t) = 1 + it \) for \( t = 1, 2, 3, \ldots \)

Note: A constant rate of simple interest does not imply a constant effective rate of interest.

\[ i_n = \]

Note: Unless stated otherwise, under simple interest it will be assumed that interest is accrued proportionally over fraction periods.

Example: A bank pays a simple interest rate of 2.5% per annum. $2,000 is deposited on January 1, 2004.

(a) Compute the accumulated value on April 1, 2006.
(b) How long until the accumulated amount is $2,230?
Example: On January 31 Bob borrows $5,000 from David and gives David a promissory note. The note states that the loan will be repaid on April 30 of the same year, with interest of 12% per annum. On March 1, David sells the promissory note to Sam, who pays David a sum of money in return for the right to collect the payment from Bob on April 30. Sam pays David an amount such that Sam’s yield (interest rate earned) from March 1 to the maturity date can be stated as an annual rate of interest of 15%.

(a) Determine the amount that Bob was to have paid David on April 30.
(b) Determine the amount that Sam paid to David and the simple interest rate David earned on an annual basis.
Section 1.5: Compound Interest

Compound interest is when the interest is automatically reinvested to earn additional interest to earn additional interest.

\[
\begin{array}{c|c}
 t & a(t) \\
0 & 1 \\
1 & 1 + i \\
2 & (1 + i) + i(1 + i) \\
3 & \\
\vdots & \\
n & \\
\end{array}
\]

The accumulation function for \( t = 1, 2, 3, \cdots \) is \( a(t) = \)

where \( i \) is the compound interest rate for the period.

Note: A constant rate of compound interest implies a constant effective rate of interest.

\[
i_n = \frac{a(n) - a(n - 1)}{a(n - 1)} =
\]

Note: Unless otherwise stated, assume that interest is accrued over fractional period for compound interest according to the formula.

Example: Find the accumulated value of $2,000 at the end of 2 years and 3 months invested at 6% per annum.
Example: Find the accumulated value of $2,000 at the end of 2 years and 3 months invested at 6% per annum. Assume a simple interest during the final fractional period. Compare this to the answer in the previous example.

Simple interest is used for mostly for transactions less than one year.

Compound interest is used almost exclusively for financial transactions covering a period of one year or more.

From this point assume compound interest in transactions over 1 year unless told otherwise.

Example: A deposit of $X is invested at time 6 years at an annual effective interest rate of 8%. A second deposit of $X is invested at time 8 years at the same interest rate. At time 11 years, the accumulated amount of the investment is $976. Calculate X.
Section 1.6: Present Value

We have seen that an investment of 1 will accumulate to $1 + i$ at the end of one period.

<table>
<thead>
<tr>
<th>account value</th>
<th>1</th>
<th>$1 + i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The term $1 + i$ is often called the accumulation factor, since it accumulates the value of an investment at the beginning of a period to its value at the end of the period.

Question: How much should be invested initially so that the balance will be 1 at the end?

<table>
<thead>
<tr>
<th>account value</th>
<th>X</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The term $v = \frac{1}{a(t)}$ is often called a discount factor, since it ”discounts” the value of an investment at the end of a period to its value at the beginning.

Find the amount that should be invested initially in order to accumulate an amount of 1 at the end of $t$ periods.

<table>
<thead>
<tr>
<th>account value</th>
<th>X</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: The book uses $a^{-1}(t)$ to denote the discount function, i.e. the reciprocal of $a(t)$. Should be $[a(t)]^{-1}$.

For simple interest: discount function = __________________________
Example: A payment of $1000 is to be made at time 7 years. The annual effective rate is 6%.

(a) Determine the present value of this payment at time 0 and at time 4.
(b) How many years will it take the account to reach $800, if the present value at time 0 is invested.
Example: An investment of $1,000 will grow to $6,000 after 20 years. Find the sum of the present values of two payments $5,000 each which will occur at the end of the 15 and 30 years, assuming the same interest rate.

Example: Mark owes you some money. He has given you two options.

Option 1: He will make a payment of $950 now and then another payment of $1000 at time 2.

Option 2: He will give you only one payment of 2000 at time 1.

What annual effective interest rate would make both options equivalent?
Section 1.7: The Effective rate of Discount

• If Sue borrows $100 from a bank for 1 year at an effective rate of interest of 5%, then at the end of one period (one year), Sue would pay back the original loan of $100 plus interest of $5 or a total of $105.

<table>
<thead>
<tr>
<th>account value</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
</tr>
</tbody>
</table>

• If Bob borrows $100 for one year at an effective rate of discount of 5%, then the bank will collects its interest of 5%, or $5, in advance and will give Bob only $95. At the end of the period, Bob will repay $100.

<table>
<thead>
<tr>
<th>account value</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Period</td>
<td>0</td>
</tr>
</tbody>
</table>

The effective rate of interest, $i$, is a measure of the interest paid at the end of the period. Or the ratio of the interest earned in the period, to the amount invested at the beginning of the period.

\[ i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} \text{ for } n = 1, 2, 3, \ldots \]

The effective rate of discount, $d$, is a measure of interest paid at the beginning of the period. Or the ratio of the amount of interest (amount of discount or just discount) earned during the period to the amount invested at the end of the period.

\[ d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)} \text{ for } n = 1, 2, 3, \ldots \]

Note: The effective rate of discount is constant for each period when compounding.
Assume a discount of \( d \) each period.

For an annual compound rate of discount, \( d \):

- The present value of a payment of $1 to be made in \( t \) years is \___________.

- The accumulated value after \( t \) years of a deposit of $1 is \___________.

Example: How much should an investor deposit today to have $4,000 in 5 years if the annual rate of discount is 6%?
Example: Compare the accumulated amount of $1000 invested for 10 years at an annual rate of interest of 6% verses an annual rate of discount of 6%.

The relationship between $d$ and $i$

Note: The following relationships are only valid for compound discount/compound interest and not for simple discount/simple interest (unless the number of periods is one).

Concept of equivalency: Two rates of interest or discounts are said to be equivalent if a given amount of principal invested for the same length of time at each of the rates produces the same accumulated value.

<table>
<thead>
<tr>
<th>Period</th>
<th>account value</th>
<th>1 - $d$</th>
<th>1</th>
<th>Period</th>
<th>account value</th>
<th>1</th>
<th>1 + $i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Simple Discount:

The amount of discount earned each period is a constant. For an annual simple rate of discount, \( d \):

- The present value of a payment of $1 to be made in \( t \) years is \( \ldots \)
- The accumulated value after \( t \) years of a deposit of $1 is \( \ldots \)

Note: simple discount is generally only used for terms less than 1 year.

Example: What is the present value of $1000 due in 10 days at a simple daily discount rate of 10%?

Example: An investment of $10,000 is made into a fund at time \( t = 0 \). The fund develops the following balances over the next 2 years. Compute the effective rate of discount for the second year.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( A(t) )</th>
<th>( I_n )</th>
<th>( i_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10,600</td>
<td>( I_1 = 600 )</td>
<td>( i_1 = 6% )</td>
</tr>
<tr>
<td>2</td>
<td>11,024</td>
<td>( I_2 = 424 )</td>
<td>( i_2 = 4% )</td>
</tr>
</tbody>
</table>

Example: Find the accumulated value of $1000 at the end of 7 years and 5 months invested at an effective rate of discount of 4% assuming simple discount in the fractional period.