MA 304 EXAM 1 ——spring 2019

REMARKS There are 8 problems. Problems 1-4 are each worth 12 points while problems 5-8 are each worth 13 points. Show all relevant work. NO CALCULATORS.

1. Given the following matrix equation

\[ w + 2x - 3y + z = 1 \]
\[ -w - x + 4y - z = 6 \]
\[ -2w - 4x + 7y - z = 1 \]

a. Put the above system into reduced row echelon form

\[
\begin{pmatrix}
1 & 2 & -3 & 1 \\
-1 & -1 & 4 & -1 \\
-2 & -4 & 7 & -1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & -3 & 1 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

b. Write an expression for all solutions to the matrix equation and write it as an equation of some plane, i.e., \((w, x, y, z)^T = \)

\[
\begin{pmatrix}
w \\
x \\
y \\
z \\
\end{pmatrix}
= \begin{pmatrix}
7 -6 \\
4 + 3 \\
3 - 9 \\
3 \\
\end{pmatrix}
\]

2. Let

\[ A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} -3 \\ -2 \end{pmatrix} \]

a. Write \(b\) as a linear combination of the columns of \(A\).

\[
\begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}
\]

b. Use the results from a. to determine a solution of the linear system \(Ax = b\).

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}
\]

c. Let \(A\) be a \(5 \times 3\) matrix. If \(a_1, a_2, a_3\) represent the columns of \(A\) and

\[ b = a_1 + a_2 = a_2 + a_3 \]

then what can you conclude about the number of solutions of the linear system \(Ax = b\)? Explain!

\[
A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b
\]

so 2 solns \(\Rightarrow\) \(\approx\) many
3. Consider the $3 \times 3$ Vandermonde matrix:

$$ V = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} $$

a. Show that $\det(V) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$.

$$ V \text{ determinant } = x_2x_3(x_2-x_1) - x_3(x_2x_3)(x_3-x_1) + x_3^2(x_2-x_1) $$

b. What conditions must the scalars $x_1, x_2, x_3$ satisfy for $V$ to be nonsingular?

$x_1, x_2, x_3$ are distinct

4. Let $A$ be the matrix

$$ A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix} $$

Find $A^{-1}$ by computing the reduced row echelon form of $A$.

$$ \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} $$

5.

a. Compute the LU factorization of

$$ A = \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -2 & 1 \end{pmatrix} $$

b. Let

$$ A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} $$

Find a $2 \times 2$ matrix $X$ which solves $AX + B = X$.
6.

a. Is the following set a spanning set for $\mathbb{R}^3$. Justify your answer.

$$\{(2, 1, -2), (3, 2, -2), (2, 2, 0)\}$$

b. Is the following a spanning set for $P_3$ (quadratic polynomials).

$$\{x^2, x^2 - 1\}$$

7.

a. Use Cramer's rule to solve the following system

$$\begin{align*}
2x + 3y &= 2 \\
3x + 2y &= 5
\end{align*}$$

$$x = \frac{12 - 3}{3 - 2} = -\frac{9}{1} = -9$$

$$y = \frac{13 - 2}{3 - 2} = -\frac{11}{1} = -11$$

b. Evaluate the determinant of the following matrix. Write you answer as a polynomial in $x$.

$$A = \begin{pmatrix}
a - x & b & c \\
1 & -x & 0 \\
0 & 1 & -x
\end{pmatrix}$$

$$(a - x)(x^2) - (-b)(x - c)$$

$$-x^3 + ax^2 + bx + c$$
8.

a. Let $x_1, x_2$ and $x_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

$$y_1 = x_2 + x_1, \quad y_2 = x_3 + x_2, \quad y_3 = x_3 + x_1$$

Are $y_1, y_2$ and $y_3$ linearly independent? Prove your answer.

\[
\begin{align*}
ay_1 + by_2 + cy_3 &= 0 \\
a(x_2 + x_1) + b(x_3 + x_2) + c(x_3 + x_1) &= 0 \\
(b + c)x_3 + (a + b)x_2 + (a + c)x_1 &= 0
\end{align*}
\]

so $b, c, 1 \in \mathbb{R}$

\[
\begin{pmatrix}
1 \\
1 \\
1
\end{pmatrix} = -2 \neq 0
\]

b. Let $x_1, x_2, \ldots, x_k$ be linearly independent vector in $\mathbb{R}^n$, and let $A$ be a nonsingular $n \times n$ matrix. Define $y_j = Ax_j$ for $j = 1, \ldots, k$. Show that $y_1, y_2, \ldots, y_k$ are linearly independent.

\[
\begin{align*}
ay_1 + \cdots + ay_k &= 0 \\
so a_1 Ax_1 + \cdots + a_k Ax_k &= 0 \\
so A(a_1 x_1 + \cdots + a_k x_k) &= 0
\end{align*}
\]

implies $a_1 x_1 + \cdots + a_k x_k = 0$ since $A$ is invertible

implies $a_1 = a_2 = \cdots = a_k = 0$

since $\{x_1, x_2, \ldots, x_k\}$ is L.I.