Section 1.2
Exercise 1.2.12

Let \( f(x) = e^x - e^{-x} \).

(a) Find \( \lim_{x \to 0} \frac{e^x - e^{-x}}{x} \).

Solution: Using L’Hospital’s Rule we get:
\[
\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = \lim_{x \to 0} \frac{e^x + e^{-x}}{1} = 2
\]

(b) Use three-digit rounding arithmetic to evaluate \( f(0.1) \);

Solution:
\[
\frac{(2.72)^{0.1} - (2.72)^{-0.1}}{0.1} = \frac{1.11 - 0.905}{0.1} = \frac{0.205}{0.1} = 2.05
\]

(c) Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).

Solution:
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]
The third degree Maclaurin polynomial is then \( \sum_{n=0}^{3} \frac{x^n}{n!} \). Our \( f(x) \) is now
\[
f(x) = \frac{\sum_{n=0}^{3} \frac{x^n}{n!} - \sum_{n=0}^{3} \frac{(-x)^n}{n!}}{x} = 2 + \frac{x^2}{3}
\]
\[
f(0.1) \approx 2 + \frac{(0.1)^2}{3} = 2.00
\]

(d) The actual value is \( f(0.1) = 2.003335000 \). Find the relative error for the values obtained in parts (b) and (c).

Solution: We are given that \( f(0.1) = 2.003335000 \).

For part (a) we have
\[
E = \left| \frac{f(0.1) - 2.05}{f(0.1)} \right| = 0.0233.
\]
For part (b) we have
\[ E = \frac{|f(0.1) - 2.00|}{|f(0.1)|} = 0.0017. \]

Section 2.1

Exercise 2.1.8

(a) Sketch the graphs of \( y = x \) and \( y = \tan(x) \).

(b) Use the bisection method to find an approximation to within \( 10^{-5} \) to the first positive value of \( x \) with \( x = \tan(x) \).

Solution: After 16 iterations we find a root at \( x = 4.49341 \).

Exercise 2.1.12

Find an approximation to \( \sqrt{3} \) correct to within \( 10^{-4} \) using the Bisection Algorithm. (Consider \( f(x) = x^2 - 3 \).)

Solution: After 14 iterations we find a root at \( x = 1.7310 \).
Exercise 2.1.14
(a) Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy $10^{-3}$ to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$.

Solution: Theorem 2.1 gives us the following error formula:

$$|p_n - p| \leq \frac{b-a}{2^n}.$$  

We want to find an $n$ such that $|p_n - p| < 10^{-3}$. This is satisfied if we bound $\frac{b-a}{2^n}$ above by $10^{-3}$. In our case $b = 4$ and $a = 1$ which gives

$$\frac{b-a}{2^n} = \frac{4-1}{2^n} < 10^{-3} \implies 3000 < 2^n \implies \log_2(3000) \approx 11.55 < n.$$  

Since $n$ is an integer, it must be at least 12.

(b) Find an approximation to the root with this degree of accuracy.

Solution: After 12 iterations we find a root at $x = 1.379$.

Exercise 2.1.18

The function defined by $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$, and $2 < b < 3$, the Bisection method converges to

(a) 0, if $a + b < 2$

Solution: The key to solving all of these problems is to look at the Bisection Algorithm on page 49 and figure out which point the algorithm will converge to. The first step is to add the two inequalities given in the problem to get $1 < a + b < 3$ and note that $p = a + \frac{b-a}{2} = \frac{a+b}{2}$.

STEP 1: $FA = \sin(\pi a) < 0$
STEP 3: $1 < a + b < 2 \implies \frac{1}{2} < p < 1$
\implies p \in (\frac{1}{2}, 1) \implies FP = \sin(\pi p) > 0$
STEP 6: $FA \cdot FP < 0 \implies b = p \in (\frac{1}{2}, 1)$
The algorithm will now search within the interval $[a, p]$ and 0 is the only root within this new interval.

(b) 2, if $a + b > 2$

Solution:

STEP 1: $FA = \sin(\pi a) < 0$
STEP 3: $3 > a + b > 2 \implies \frac{3}{2} > p > 1$
\implies p \in (1, \frac{3}{2}) \implies FP = \sin(\pi p) < 0$
STEP 6: $FA \cdot FP > 0 \implies a = p \in (1, \frac{3}{2})$
The algorithm will now search within the interval $[p, b]$ and 2 is the only root within this new interval.

(c) 1, if $a + b = 2$

Solution:

STEP 1: $FA = \sin(\pi a) < 0$
STEP 3: $a + b = 2 \implies p = 1$
\implies FP = \sin(\pi p) = 0$

3
STEP 4: $FP = 0$
The algorithm will terminate with $p = 1$ as the root.

Section 2.2

Exercise 2.2.4
The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

(a) $p_n = p_{n-1} \left(1 + \frac{7 - p_n^5}{p_{n-1}^5} \right)^3$

Solution: Diverges after 3 iterations with Tol = $10^{-3}$

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>$p_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>343</td>
</tr>
<tr>
<td>2</td>
<td>$-2.25 \times 10^{25}$</td>
</tr>
<tr>
<td>3</td>
<td>$3.38 \times 10^{25}$</td>
</tr>
</tbody>
</table>

(b) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

Solution: Diverges after 5 iterations with Tol = $10^{-3}$

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>$p_n$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>$-335$</td>
</tr>
<tr>
<td>3</td>
<td>$3.78 \times 10^7$</td>
</tr>
<tr>
<td>4</td>
<td>$-5.44 \times 10^{22}$</td>
</tr>
<tr>
<td>5</td>
<td>$1.6 \times 10^{68}$</td>
</tr>
</tbody>
</table>

(c) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$

Solution: Converges after 6 iterations with Tol = $10^{-3}$

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>$p_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.2000</td>
</tr>
<tr>
<td>2</td>
<td>1.8198</td>
</tr>
<tr>
<td>3</td>
<td>1.5835</td>
</tr>
<tr>
<td>4</td>
<td>1.4895</td>
</tr>
<tr>
<td>5</td>
<td>1.4760</td>
</tr>
<tr>
<td>6</td>
<td>1.4758</td>
</tr>
</tbody>
</table>

(d) $p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$

Solution: Converges after 163 iterations with Tol = $10^{-3}$. Note the values jumping around in the table and not settling down.
<table>
<thead>
<tr>
<th>Iteration #</th>
<th>$p_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.5000</td>
</tr>
<tr>
<td>2</td>
<td>1.4505</td>
</tr>
<tr>
<td>3</td>
<td>1.4987</td>
</tr>
<tr>
<td>4</td>
<td>1.4519</td>
</tr>
<tr>
<td>5</td>
<td>1.4976</td>
</tr>
<tr>
<td>6</td>
<td>1.4532</td>
</tr>
<tr>
<td>7</td>
<td>1.4965</td>
</tr>
</tbody>
</table>

Exercise 2.2.8

(a) Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$.

Solution: $g : [\frac{1}{3}, 1] \rightarrow [g(1), g(\frac{1}{3})] \approx [0.5, 0.7937] \subset [\frac{1}{3}, 1]$, so (i) of Theorem 2.3 is satisfied. To confirm (ii) of Theorem 2.3 we notice that $g'(x) = -\ln(2)2^{-x} : (\frac{1}{3}, 1) \rightarrow -\ln(2) * (g(1), g(\frac{1}{3})) \approx -\ln(2) * (0.5, 0.7937) \approx (-0.5502, -0.3466) \implies \max_{x \in (\frac{1}{3}, 1)} |g'(x)| < 0.5502 < 1$. So (ii) of Theorem 2.3 is satisfied and $g(x)$ must have a fixed point on the interval $[\frac{1}{3}, 1]$.

(b) Use fixed-point iteration to find an approximation to the fixed point accurate to within $10^{-4}$.

Solution: Using the fixed-point algorithm we get a fixed point at $x = 0.6411$ after 11 iterations.

(c) Use Corollary 2.5 to estimate the number of iterations required to achieve $10^{-4}$ accuracy, and compare this theoretical estimate to the number actually needed.

Solution: Using part (a) we see that $\max_{x \in (\frac{1}{3}, 1)} |g'(x)| < 0.5502$. Let’s take $k = 0.5502$ in Corollary 2.5. We want $|p_n - p| < 10^{-4}$. This will be true if $k^n \max\{p_0 - a, b - p_0\} < 10^{-4}$. Let’s take $p_0 = 1 = b$ to make the calculations simpler. We now have

$$k^n(1 - \frac{1}{3}) = k^n \frac{2}{3} < 10^{-4} \implies k^n < \frac{\ln(0.00015)}{\ln(0.5502)} \approx 14.74.$$ 

So $n > 15$. OH NO! Our estimate for $n$ seems to be greater than the actual number of iterations it took in part (b) (11 iterations), what happened? If we look at the result of part (b) we see that the fixed point is near 0.64. Taking a tighter interval of $[0.6, 0.7]$ and performing the same analysis as above we find this gives us a lower bound on $n$ of 9. The lack of accuracy in our initial guess of an interval $[\frac{1}{3}, 1]$ is limiting our ability to analyze the problem.

Exercise 2.2.10

(a) Use a fixed-point iteration method to find an approximation to $\sqrt{25}$ that is accurate to within $10^{-4}$.

Solution: We need to first derive a function $g(x)$ to use in the fixed-point algorithm. $\sqrt{25}$ satisfies the equation $25 = x^3$ and we want something like $x = g(x)$.

$$x^3 - 25 = 0 \implies x = x + 0 = x + (x^3 - 25) = g(x)$$

This is a good initial guess. Let’s see if $g(x)$ satisfies the conditions of Theorem 2.3 or if we can maybe get some clues from Theorem 2.3 about how to construct $g(x)$. Looking at $g'(x) = 1 + 3x^2$ on the interval $[2, 3]$ (which contains $\sqrt{25}$), we see that $|g'(x)| > 1$ and so (ii) of Theorem 2.3 is not satisfied. We need to scale the $3x^2$ part of $g'(x)$ so that it will take away from the 1 portion of it. Let’s look at $g_1(x) = x + \mu(x^3 - 25)$. This has derivative $g_1'(x) = 1 + 3x^2$ which can be scaled to less than 1 on the interval $[2, 3]$ by choosing an appropriate $\mu$, say $\mu = -\frac{1}{27}$. We therefore use the function

$$g(x) = x - \frac{1}{27}(x^3 - 25)$$
in our fixed point algorithm. Using an initial guess of $p_0 = 3$ we get the following results:

| Iteration # | $p_n$  | $|p - p_n|$ |
|-------------|--------|-----------|
| 1           | 2.9259 | 0.001908  |
| 2           | 2.9241 | 0.000094  |
| 3           | 2.9240 | 0.000005  |

(b) Compare your result and the number of iterations required with the answer obtained in Exercise 2.1.13.

**Solution:** Exercise 2.1.13 took 14 iterations to complete (according to the solutions in the back of the book) while the fixed-point method took 3 iterations. This seems to indicate that the fixed-point method is faster.