Section 11.3: Nonlinear Autonomous Systems (Theory)

In this section, we will study autonomous first-order systems of differential equations, that is, systems of the form

\[
\begin{align*}
\frac{dx_1}{dt} &= f(x_1, x_2) \\
\frac{dx_2}{dt} &= g(x_1, x_2)
\end{align*}
\]

where the functions \( f, g \) are real-valued and do not explicitly depend on \( t \). We will assume at least one of \( f, g \) is nonlinear, giving a nonlinear system. This system can be expressed in matrix form as

\[
\begin{align*}
\frac{dx}{dt} &= F(x)
\end{align*}
\]

where \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) and \( F(x) = \begin{bmatrix} f(x_1, x_2) \\ g(x_1, x_2) \end{bmatrix} \). A solution to this system is a pair of functions \( x_1(t), x_2(t) \) that satisfies both differential equations above.

In general, it is not possible to find explicit solutions of a nonlinear system of differential equations, but we can still analyze equilibria and their stability.

An equilibrium of a nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= F(x)
\end{align*}
\]

is a solution \( \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \) for which there is no change, that is,

\[
\begin{align*}
\frac{d\hat{x}}{dt} &= F(\hat{x}) = 0
\end{align*}
\]

An equilibrium is called **locally stable** if solutions that begin close to the equilibrium approach it, and **unstable** if solutions that begin close to the equilibrium move away from it.
Stability (Analytical Approach):

**Hartman-Grobman Theorem (Linearization Theorem):** Suppose \( \hat{x} \) is an equilibrium of

\[
\frac{dx}{dt} = F(x)
\]

such that the Jacobi matrix \( J(\hat{x}) \) has eigenvalues with nonzero real part. Then the stability of \( \hat{x} \) can be determined from the linearized system

\[
\frac{dx}{dt} = J(\hat{x})x
\]

*Note:* This theorem states that the vector field of the linearized system is similar to the vector field of the nonlinear system near \( \hat{x} \), provided the eigenvalues of \( J(\hat{x}) \) have nonzero real part. We can then extend the classification for linear systems (stable node, unstable node, etc.) to equilibria of nonlinear systems, but we have to be careful to conclude *locally* stable when necessary.

**Example 1:** Consider the nonlinear system

\[
\frac{dx_1}{dt} = -2 \sin(x_1)
\]
\[
\frac{dx_2}{dt} = -x_2 e^{x_1}
\]

Show that \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) is an equilibrium and determine its stability.
Example 2: Find all equilibria of the nonlinear system

\[
\frac{dx_1}{dt} = x_1 x_2 - x_2 \\
\frac{dx_2}{dt} = x_1 + x_2
\]

and determine the stability of each equilibrium.
Example 3: Find all equilibria of the nonlinear system

\[
\frac{dx_1}{dt} = 2x_1(5 - x_1 - x_2)
\]
\[
\frac{dx_2}{dt} = 3x_2(7 - 3x_1 - x_2)
\]

and determine the stability of each equilibrium.
(Example 3 continued)
Stability (Graphical Approach):
Consider a nonlinear system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= f(x_1, x_2) \\
\frac{dx_2}{dt} &= g(x_1, x_2)
\end{align*}
\]

The curves in the \(x_1x_2\)-plane defined by \(f(x_1, x_2) = 0\) and \(g(x_1, x_2) = 0\) are called the zero isoclines (or null clines) of the system. These curves represent points in the \(x_1x_2\)-plane where either \(\frac{dx_1}{dt} = 0\) or \(\frac{dx_2}{dt} = 0\). Therefore, a point of intersection of the zero isoclines is an equilibrium. We can use zero isoclines to approximate a direction field for the system, and thus sometimes classify equilibria.

(Motivational) Example 4: Consider the system

\[
\begin{align*}
\frac{dx_1}{dt} &= 10 - 2x_1 - x_2 \\
\frac{dx_2}{dt} &= 10 - x_1 - 2x_2
\end{align*}
\]

(a) Find the equations of the zero isoclines of the system.

(b) Find all equilibria of the system.
The zero isoclines and equilibrium are graphed below. The $f = 0$ isocline is in blue and the $g = 0$ isocline is in red. Note that the equilibrium subdivides the isoclines into 4 pieces.

We can approximate a direction field using the following:

1. For a point $(x_1, x_2)$ on the blue $f = 0$ isocline, we have $\frac{dx_1}{dt} = 0$, so the tangent vector is of the form $\begin{bmatrix} 0 \\ \frac{dx_2}{dt} \end{bmatrix}$, i.e., is vertical (it points up if $\frac{dx_2}{dt} = g(x_1, x_2) > 0$ and down if $\frac{dx_2}{dt} = g(x_1, x_2) < 0$). The behavior along a piece is the same by continuity.

2. For a point $(x_1, x_2)$ on the red $f = 0$ isocline, we have $\frac{dx_2}{dt} = 0$, so the tangent vector at such a point is of the form $\begin{bmatrix} \frac{dx_1}{dt} \\ 0 \end{bmatrix}$, i.e., is horizontal (it points right if $\frac{dx_1}{dt} = f(x_1, x_2) > 0$ and left if $\frac{dx_1}{dt} = f(x_1, x_2) < 0$). The behavior along a piece is the same by continuity.

3. Use the arrows from (1) and (2) to draw an arrow in each region bounded by isoclines.