Section 12.6: Cylinders and Quadric Surfaces

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve. We will focus on two-variable equations, where rulings are parallel to a coordinate axis.

Example 1: Sketch the following surfaces.

(a) **elliptic cylinder**: \( \frac{y^2}{4} + \frac{z^2}{16} = 1 \)

(b) **parabolic cylinder**: \( z = x^2 \)

(c) **hyperbolic cylinder**: \( \frac{y^2}{9} - x^2 = 1 \)
Quadric Surfaces: A quadric surface is the graph of a second-degree equation in three variables. They are characterized by the curves of intersection of the surface with planes of the form $x = k$, $y = k$, and $z = k$. These curves are called traces (or cross-sections) of the surface.

Ellipsoid: The quadric surface defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an ellipsoid. All traces are ellipses.

Hyperboloid of one sheet: A quadric surface defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{or} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called a hyperboloid of one sheet. The variable with a negative coefficient determines the axis that the hyperboloid opens up along. Traces are hyperbolas and ellipses.
Hyperboloid of two sheets: A quadric surface defined by
\[
-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1 \quad \text{or} \quad -x^2/a^2 + y^2/b^2 - z^2/c^2 = 1 \quad \text{or} \quad x^2/a^2 - y^2/b^2 - z^2/c^2 = 1
\]
is called a hyperboloid of two sheets. The variable with a positive coefficient determines the axis that the hyperboloid opens up along. Traces are hyperbolas and ellipses.

(Elliptic) Cone: A quadric surface defined by
\[
\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{or} \quad \frac{y^2}{b^2} = \frac{z^2}{c^2} + \frac{x^2}{a^2} \quad \text{or} \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} + \frac{z^2}{c^2}
\]
is called an (elliptic) cone. The variable by itself on one side of the equal sign determines the axis that the cone opens up along. Traces are hyperbolas/lines and ellipses.
Elliptic Paraboloid: A quadric surface defined by

\[ \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{or} \quad \frac{y}{b} = \frac{x^2}{a^2} + \frac{z^2}{c^2} \quad \text{or} \quad \frac{x}{a} = \frac{y^2}{b^2} + \frac{z^2}{c^2} \]

is called an elliptic paraboloid. The linear variable determines the axis that the graph opens up along, and its coefficient’s sign determines the direction. Traces are ellipses and parabolas.

Hyperbolic Paraboloid: A quadric surface defined by

\[ \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \text{or} \quad \frac{y}{b} = \frac{x^2}{a^2} - \frac{z^2}{c^2} \quad \text{or} \quad \frac{x}{a} = \frac{y^2}{b^2} - \frac{z^2}{c^2} \]

is called a hyperbolic paraboloid. The surface is “saddle-shaped” and “opens up” according to the two variables with the same sign (one always linear). Traces are hyperbolas and parabolas.
Example 2: Sketch the following quadric surfaces. Label any intercepts with the coordinate axes.

(a) \(36x^2 - 9y^2 - 4z^2 = 36\)

(b) \(\frac{x^2}{16} + \frac{y^2}{4} = 1 + z^2\)

(c) \(y = x^2 + z^2\)