A Brief Introduction to Tensors*

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1 Preliminaries

In general, a tensor is a multilinear transformation defined over an underlying finite dimensional vector space. In this brief introduction, tensor spaces of all integral orders will be defined inductively. Initially the underlying vector space, \( V \), will be assumed to be an inner product space in order to simplify the discussion. Subsequently, the presentation will be generalized to vector spaces without inner product. Usually, bold-face letters, \( a \), will denote vectors in \( V \) and upper case letters, \( A \), will denote tensors. The inner product on \( V \) will be denoted by \( a \cdot b \).

2 Tensors over an Inner Product Space

First tensor spaces are developed over an inner product vector space \( \{ V, \cdot \} \). Implicit in the discussion is use of an orthonormal basis in \( V \) for deriving component representations of vectors and tensors.

**Zero-Order Tensors.** The space of Zero-Order Tensors, \( T^0 \), is isomorphic to the scalar field, \( \mathcal{F} \), corresponding to the underlying vector space \( V \), which in this course will be either the real or complex numbers, \( \mathbb{R} \) or \( \mathbb{C} \), respectively. A zero order tensor, \( \alpha \in T_0 \), acts as a linear transformation from \( T^0 \to T^0 \)

\[
\alpha[\cdot] : T^0 \to T^0
\]

via multiplication of scalars. That is, given \( \beta \in T_0 \),

\[
\beta \mapsto \alpha[\beta] := \alpha \beta.
\]

**First-Order Tensors.** The space of First-Order Tensors, \( T^1 \), is isomorphic to the underlying vector space, \( V \). A first order tensor, \( a \in T^1 \), acts as a linear transformation from \( T^0 \to T^1 \) and from \( T^1 \to T^0 \) as follows. In the first instance, for all \( \alpha \in T^0 \),

\[
a[\alpha] := \alpha a
\]

whereas in the second instance, for all \( b \in T^1 \),

\[
a[b] := a \cdot b.
\]

It should be noticed, that both (1) and (2) are bi-linear forms, i.e. they are linear forms in each of their two independent variables separately.

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Second-Order Tensors. The space of Second-Order Tensors, $\mathcal{T}^2$, is isomorphic to the space, $\text{Lin}[V]$, of linear transformations $A : V \rightarrow V$. Elements in $\mathcal{T}^2$ act as linear transformations from $\mathcal{T}^i \rightarrow \mathcal{T}^j$ with $i, j = 0, 1, 2$ subject to $i + j = 2$. For $i = 0, j = 2$ one has for $A \in \mathcal{T}^2$ and $\alpha \in \mathcal{T}^0$

$$A[\alpha] := \alpha A. \quad (3)$$

For $i = 1, j = 1$, one has for $A \in \mathcal{T}^2$ and $a \in \mathcal{T}^1$

$$A[a] := Aa \quad (4)$$

where the expression of the right hand side of (4) denotes the action of the linear transformation $A \in \text{Lin}[V]$ on the vector $a \in V$. For $i = 2, j = 0$, one has for $A, B \in \mathcal{T}^2$

$$A[B] := A \cdot B \quad (5)$$

where $A \cdot B$ denotes the natural inner product on $\text{Lin}[V]$ defined by

$$A \cdot B := \text{tr}[A^TB]. \quad (6)$$

In (6), $\text{tr}[A]$ denotes the trace of $A \in \text{Lin}[V]$ and $A^T$ denotes the transpose of $A$.

Finally, a second-order tensor $A \in \mathcal{T}^2$ can be used to define a bi-linear transformation on $V$. Specifically, $A[\cdot , \cdot] : \mathcal{T}^1 \times \mathcal{T}^1 \rightarrow \mathcal{T}^0$ is defined by

$$\langle a, b \rangle_A := a \cdot (A[b]) \quad (7)$$

for all $a, b \in \mathcal{T}^1$.

An important class of second order tensors is given by the Elementary Tensor Product of two first order tensors. Specifically, given $a, b \in \mathcal{T}^1$ the elementary tensor product $a \otimes b$ of $a$ and $b$ is the second order tensor whose action on a first order tensor $c \in \mathcal{T}^1$ is defined by

$$a \otimes b[c] := a(b \cdot c). \quad (8)$$

From the definitions (3), (4), one sees that

$$[a \otimes b]^T = b \otimes a$$

$$\text{Tr}[a \otimes b] = a \cdot b$$

$$a \otimes b \cdot c \otimes d = (a \cdot c)(b \cdot d)$$

$$a \otimes bc \otimes d = (b \cdot c)a \otimes d.$$

Coordinates with Respect to an Orthonormal Basis. Given an orthonormal basis, $\{e_1, \ldots, e_n\}$, for the underlying vector space $V$, one can construct a “natural” orthonormal basis for the space of second order tensors, $\mathcal{T}^2$, of the form

$$\{e_i \otimes e_j, i, j = 1, \ldots, n\}. \quad (9)$$

Consequently, given $A \in \mathcal{T}^2$, one has

$$A = \sum_{i,j} a^{ij}e_i \otimes e_j$$
with the “coordinates” of $A$ relative to the natural basis given by

$$a^{ij} = A \cdot e_i \otimes e_j = e_i \cdot (Ae_j).$$

Hence, if $x \in V$ has coordinates $x^k$ relative to the basis $\{e_1, \ldots, e_n\}$, then the action of $A$ on $x$ can be computed using components

$$[Ax] = [a^{ij}x^j].$$

where summation over the index $j$ is implied. It is useful to note that one easily shows that if $a, b \in V$ have coordinates $[a] = [a^i]$ and $[b] = [b^i]$, respectively, relative to the orthonormal basis $\{e_1, \ldots, e_n\}$, then the components of the second-order tensor $a \otimes b$ relative to the natural basis on $\mathcal{T}^2$ are

$$[a \otimes b] = [a^i b^j].$$

Finally, the component form of the bi-linear form (10) is

$$\langle a, b \rangle_A = A \cdot a \otimes b = a \cdot (Ab) = a^{ij}a^i b^j$$

where summation over $i, j = 1, \ldots, n$ is implied.

**Third-Order Tensors.** The space of third-order tensors, $\mathcal{T}^3$, is most easily constructed by first considering elementary tensor products of the form $a \otimes b \otimes c$ for first-order tensors (vectors in $V$) $a, b, c \in \mathcal{T}^1$. A third-order tensor can be used to define a linear transformation from $\mathcal{T}^p \rightarrow \mathcal{T}^{3-p}$ for $p = 0, 1, 2, 3$. The action of a third-order elementary tensor product as such a linear transformation can be completely specified by defining its action on $p^{th}$ order elementary tensor products. (Why?) For $p = 0$ and $\alpha \in \mathcal{T}^0$, one defines

$$a \otimes b \otimes c[\alpha] := a\alpha \otimes b \otimes c \in \mathcal{T}^3.$$  

For $p = 1$ and $d \in \mathcal{T}^1$, one defines

$$a \otimes b \otimes c[d] := (c \cdot d)a \otimes b \in \mathcal{T}^2.$$  

For $p = 2$ and $d \otimes e \in \mathcal{T}^2$, one defines

$$a \otimes b \otimes c[d \otimes e] := (b \cdot d)(c \cdot e)a \in \mathcal{T}^1.$$  

It should be noted that in this expression, the scalar multiplying $a$ can also be written as $b \otimes c \cdot d \otimes e$, where “$\cdot$” now denotes the dot-product on $\mathcal{T}^2$ defined previously.

Finally, for $d \otimes e \otimes f \in \mathcal{T}^3$, one defines

$$a \otimes b \otimes c[d \otimes e \otimes f] := (a \cdot d)(b \cdot e)(c \cdot f).$$  

(10)  

Note that this last expression is a multi-linear form on the six variables $a, \ldots, f$. Moreover, this last expression can be used to define an inner product on the space $\mathcal{T}^3$. Indeed, as a natural basis for $\mathcal{T}^3$ one takes the set of elementary tensor products, $\{e_i \otimes e_j \otimes e_k, i, j, k = 1, \ldots, n\}$, and then defines the dot-product on $\mathcal{T}^3$ using (10) to define

$$e_i \otimes e_j \otimes e_k \cdot e_l \otimes e_m \otimes e_n := (e_i \cdot e_l)(e_j \cdot e_m)(e_k \cdot e_n) = \delta_{il}\delta_{jm}\delta_{kn}$$

where $\delta_{ij}$ denotes the Kronecker symbol

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j \end{cases}.$$
and extending the definition to all of $\mathcal{T}^3$ by linearity. In particular, a general third order tensor $A \in \mathcal{T}^3$ has the component representation $A = [a_{ijk}]$ where the $a_{ijk}$ are define through

$$A = a_{ijk} e_i \otimes e_j \otimes e_k$$

where summation over $i, j, k$ is implied.

**Fourth and Higher-Order Tensors.** The generalization to fourth-order tensors and higher should now be clear. One first defines the special class of $N^{th}$-order elementary tensor products of first-order tensors, and then uses the dot product to define their various actions as multi-linear transformations. The vector space of all $N^{th}$ or tensors is then constructed by taking all finite linear combinations of such $N^{th}$ order elementary tensor products.

For example, an $N^{th}$ order tensor elementary tensor product of the form

$$A = a_1 \otimes \ldots \otimes a_{N-p} \otimes b_1 \otimes \ldots \otimes b_p$$

defines a multilinear transformation $A : \mathcal{T}_p \rightarrow \mathcal{T}_{N-p}$ through

$$A[c_1 \otimes \ldots \otimes c_p] = a_1 \otimes \ldots \otimes a_{N-p}(b_1 \cdot c_1) \ldots (b_p \cdot c_p).$$

An important fourth-order tensor in applications is the *Elasticity Tensor* of linear elasticity theory. Specifically, the elasticity tensor, $D$, is the fourth-order tensor by which the stress tensor, $T$, is computed from the infinitesimal strain tensor, $E$, as

$$T = D[E]$$

which in component form becomes

$$t^{ij} = d^{ijkl} e_{kl}$$

where summation over $k, l = 1, 2, 3$ is implied.

### 3 Tensors over a Vector Space without Inner Product

The construction of tensor spaces of all orders given below proceeds in somewhat the same fashion as done previously, only now the underlying vector space, $\mathcal{V}$, is not assumed to have an inner product. In particular, the term orthonormal basis has no meaning in this context. However, many of the conveniences of an orthonormal basis can be realized through the introduction of the notion of a *Dual Space* to $\mathcal{V}$.

#### 3.1 Dual Space and Dual Basis

##### 3.1.1 Dual Space

Given a finite dimensional vector space $\mathcal{V}$, one defines its *Dual Space* $\mathcal{V}^*$ to be $\text{Lin}[\mathcal{V}, \mathbb{R}]$, the vector space of all linear transformations from $\mathcal{V}$ to the real numbers (or more generally, to the associated scalar field $\mathcal{F}$). Recall that if $\mathcal{V}$ has dimension $N$, then $\text{Lin}[\mathcal{V}, \mathbb{R}]$ can be realized as all $1 \times N$ matrices with real entries. The *action* of a linear transformation $a^* \in \mathcal{V}^*$ on a vector $b \in \mathcal{V}$ is denoted by $\langle a^*, b \rangle$.

**Example.** Let $\mathcal{V} = \mathbb{R}^N$ (ignoring its natural inner product). Elements $a \in \mathcal{V}$ are $n$-tuples of real numbers

$$a = \begin{pmatrix}
a^1 \\
\vdots \\
a^N
\end{pmatrix}$$
whereas elements $b^* \in V^*$ are $1 \times N$ matrices

$$b^* = \begin{pmatrix} b_1 & \ldots & b_N \end{pmatrix}.$$ 

The action $\langle b^*, a \rangle$ is then given by matrix multiplication

$$\langle b^*, a \rangle = \begin{pmatrix} b_1 & \ldots & b_N \end{pmatrix} \begin{pmatrix} a^1 \\ \vdots \\ a^N \end{pmatrix} = b_1 a^1 + \ldots + b_N a^N.$$ 

3.1.2 Dual Basis

Given a basis $B = \{e_1, \ldots, e_N\}$ for $V$, one defines its Dual Basis to be the unique basis $B^* = \{e^1, \ldots, e^N\}$ for the dual space satisfying

$$\langle e^i, e_j \rangle = \delta^i_j, \quad \text{for } i, j = 1, \ldots, N \quad (11)$$

where $\delta^i_j$ denotes the Kronecker symbol. Every vector $a \in V$ has a representation $a = a^1 e_1 + \ldots + a^N e_N$. The coefficients $a^i$, $i = 1, \ldots, N$ are called the Contravariant Coordinates of the vector $a \in V$. Correspondingly, every dual vector $b^* \in V^*$ has a representation $b^* = b_1 e^1 + \ldots + b_N e^N$, with the coefficients $b_i$, $i = 1, \ldots, N$ being called the Covariant Coordinates of $b^*$. It follows from (11) that

$$a^i = \langle e^i, a \rangle \quad \text{and} \quad b_i = \langle b^*, e_i \rangle. \quad (12)$$

3.2 The Tensor Spaces

The tensor space $T^p_q(V)$ is defined to be the vector space of all $(p+q)$-multilinear, real-valued functions

$$A : \underbrace{V^* \times \ldots \times V^*}_{\text{p-times}} \times \underbrace{V \times \ldots \times V}_{\text{q-times}} \to \mathbb{R}. \quad (13)$$

Thus, $A$ is a function of $p$-variables from $V^*$ and $q$-variables from $V$ that is linear in each variable separately. The Contravariant Order of $A$ is $p$ and the Covariant Order of $A$ is $q$. A Pure Contravariant Tensor has order $(p, 0)$ while a Pure Covariant Tensor has order $(0, q)$.

Example. Every transformation $A \in \text{Lin}(V)$ defines a tensor $\hat{A} \in T^1_1$ through

$$\hat{A}(v^*, v) = \langle v^*, Av \rangle$$

for every $v^* \in V^*$ and $v \in V$.

Example. Any $p$-vectors from $V$ and $q$-dual vectors from $V^*$ can be used to construct a tensor in $T^p_q$ in the form of a tensor product. More specifically, if $v_1, \ldots, v_p \in V$ and $v^1, \ldots, v^q \in V^*$, then one defines the tensor product $v_1 \otimes \ldots v_p \otimes v^1 \otimes \ldots v^q \in T^p_q$ through the action

$$v_1 \otimes \ldots v_p \otimes v^1 \otimes \ldots v^q(u_1, \ldots, u^p, u_1, \ldots, u^q) \rightarrow \langle u^1, v_1 \rangle \cdots \langle u^p, v_p \rangle \times \langle v^1, u_1 \rangle \cdots \langle v^q, u^q \rangle.$$ 

Given a basis $B = \{e_1, \ldots, e_N\}$ for $V$ with associated dual basis $B^* = \{e^1, \ldots, e^N\}$ for $V^*$, one constructs the natural product basis for $T^p_q$ as

$$\{e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e^{j_1} \otimes \ldots \otimes e^{j_q}, i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, N\}.$$
Thus, one sees the \( \dim(T^p_q) = N^{(p+q)} \) where \( \dim(V) = N \). It is now straightforward to construct the component form of a general tensor. Hence, for a tensor \( A \in T^p_q \), one defines its component form relative to the natural product basis

\[
[A] = \begin{bmatrix} a^i_{j_1 \ldots j_q} \end{bmatrix}
\]

through the following argument. Let \( u^1, \ldots, u^p \in V^* \) have covariant coordinates \( [u^i]_{B^*} = [u^i_k]_B, i = 1, \ldots, p, \ k = 1, \ldots, N \) and let \( u_1, \ldots, u_q \in V \) have contravariant coordinates \( [u_j]_B = [u^k_j], j = 1, \ldots, q, \ k = 1, \ldots, N \). Then,

\[
A(u^1, \ldots, u^p, u_1, \ldots, u_q) = a^i_{j_1 \ldots j_q} e^i_{j_1} \otimes \ldots \otimes e^i_{j_p} \otimes e^{j_1} \otimes \ldots \otimes e^{j_q}(u^1, \ldots, u^p, u_1, \ldots, u_q)
\]

\[
= a^i_{j_1 \ldots j_q} \langle u^1, e_{i_{j_1}} \rangle \ldots \langle u^p, e_{i_{j_p}} \rangle \langle e^{j_1}, u_1 \rangle \ldots \langle e^{j_q}, u_q \rangle
\]

\[
= a^i_{j_1 \ldots j_q} u^1_{i_1} \ldots u^p_{i_p} u^1_{j_1} \ldots u^q_{j_q}.
\]

**Generalized Tensor Product.** There is a useful generalization of the elementary tensor product to tensors of arbitrary order. Specifically, given \( A \in T^p_q \) and \( B \in T^r_s \) with \( r \leq p \) and \( s \leq q \), one then defines the dot product \( A \cdot B \) to be a tensor in \( T^{p-r}_{q-s} \) given (in component form) by

\[
[A \cdot B] = \begin{bmatrix} a^i_{j_1 \ldots j_q} b^j_{q-s+1 \ldots j_q} \end{bmatrix}
\]

Thus, the orders of tensors subtract in forming tensor products.

**Generalized Contraction.** Similarly, it is useful to introduce a generalization of the dot product (contraction operator) to higher order tensors. To that end, let \( A \in T^p_q \) and \( B \in T^r_s \) with \( r \leq p \) and \( s \leq q \). One then defines the dot product \( A \cdot B \) to be a tensor in \( T^{p-r}_{q-s} \) given (in component form) by

Thus, the orders of tensors subtract in the generalized dot product.