Exponential, Logistic, and Logarithmic Functions

**Topic Sequence**

1. Laws/Properties of Exponents and Simplifying Exponential Expressions
2. Solving Exponential Equations
3. Graphing Exponential Functions
4. Laws/Properties of Logarithms and Simplifying Exponential Expressions
5. Solving Logarithmic Equations
6. Graphing Logarithmic Functions
7. Properties of Logistic Functions
8. Modeling with Exponential, Logarithmic, and Logistic Functions (note: I would cover specific modeling questions in each of sections 2, 5, and 7 as well – section 8 would simply be a synopsis of the different contexts in which those models are used)

**Lesson 7: Properties of Logistic Functions**

By now, students have covered the inner workings of exponential and logarithmic functions. We now introduce them to a class of functions called “logistic” functions, and we do so using the following activity. Important questions to ask the students are labeled in **bold type**, and teacher thoughts and comments follow in *italicized type*. Be sure to read through the entire activity before you try it with your students! (There are a couple sample simulations at the end of the lesson for reference.)

Upon completion of this exploratory activity, we would continue studying and solidifying the features and uses of logistic functions.

**Logistic Function Introduction**

The students are going to model the spread of a very contagious disease. For simplicity sake, let’s suppose that each day, a person can pass the disease to one other person through mere proximity (and since we’ll be in the same classroom for days, proximity won’t be a problem).

The directions are relatively simple:

1. Have the students number off 1 through however many there are – you can include yourself if you choose. Make sure they remember their numbers!

2. Have the students copy the table below, adding rows as needed (10-15 rows should be adequate):

<table>
<thead>
<tr>
<th>Day</th>
<th>No. of people infected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
(3) Have all the students stand up at their desks (or in some other clever formation, if you so desire). They represent the healthy individuals. Once they are stricken with the disease, they can sit down.

(4) Use the random integer generating function on your calculator to pick a number between 1 and \( n \), where \( n \) is the number of people that numbered off (let’s assume that number is 30). On the TI-83/83+/84 calculators, this function is under \( \text{MATH} \rightarrow \text{PRB} \rightarrow \#5:\text{randInt} \) and it requires two arguments \( a \) and \( b \), where we are picking a number between \( a \) and \( b \), inclusive. This integer represents the number of the person who first contracted the disease (and you can make up your own story about how/where/why/when/etc.) Have that person sit down. They are now infected.\(^1\) Now there is one person infected, who will infect one other person tomorrow (after one day). Fill in the first line of the table so that it reads as follows.

<table>
<thead>
<tr>
<th>Day</th>
<th>No. of people infected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

(5) After one full day, a second person will be infected, and so we need to select another person at random, so use the randInt function again and have that person sit down. Write the total number of distinct disease-carriers in the table so that the table now reads as shown below. **What happens if we get the same person’s number again? How could that occurrence be interpreted in light of the actual “natural” situation?** (Of course, there is a small chance of this happening, but it COULD.) Possible explanations: The person didn’t come in contact with anyone, the disease needs time to incubate, etc.

<table>
<thead>
<tr>
<th>Day</th>
<th>No. of people infected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

(6) Ask the students: **How many people will be infected after two days?** This SHOULD be 4 (the first person, the person infected after one day, and the two people infected by the first two. Use the randInt function to determine the new carriers of the disease.\(^2\) However, it COULD happen that one of the numbers that was already chosen is selected again. In this case, ask: **How many NEW people were infected after two days?** This will be either 3 or 4. If the

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\(^1\) The syntax to generate an integer between 1 and 30 is “randInt(1,30)”.  
\(^2\) The syntax for generating \( m \) random numbers between \( a \) and \( b \) is randInt(\( a,b,m \)), or in this case, randInt(1,30,2).
If the result was 4, then ask the students if they can figure out what mathematical model fits the data so far (exponential, namely \( y = 2^x \), and go on to the next step. If the result was 3, then discuss what this could mean in terms of the spread of the disease, i.e., the infected person only came in contact with someone who was already infected, the person who was already infected is now immune, etc. In either case, put the total number of distinct disease-carriers in the table so that the table reads either 3 or 4 for day 2.

(7) How many people do you think will be infected after three days? If the answer to step 6 was 4, then this SHOULD be 8, but the further we go into this simulation, the more likely it will be that the data will no longer fit an exponential model. If the answer to step 6 was 3, then the students should already suspect something different and might be unsure how to predict the result of the next step. Try to lead them to ask themselves the question “what SHOULD have happened?” and then “why DIDN’T that happen (in terms of the nature of passage of a disease)?”

(8) Continue in this fashion until all members of the class are infected, using the total number of people infected on day \( n \) as the number of random numbers you generate on day \( n + 1 \) (why?), and filling in the table as you go.

(9) Once the table is complete, enter the days 0, 1, \ldots, \( n \) into L1 on your calculator and the corresponding total number infected into L2. Then set your “Stat plot” settings as shown below, and press ZOOM:9(zoomdata). You should get a plot similar to the one shown below.

(10) Note that this curve is not purely exponential, nor is it purely logarithmic. Instead, it is what we call “logistic.” We will now use the regression features of our calculator to determine its equation (which would be difficult to do by hand). From the home screen, go to \text{STAT} \rightarrow \text{CALC} \rightarrow \text{B:Logistic} and press ENTER. This brings you to the home screen with Logistic and the cursor flashing afterwards. Now complete the command so that it reads “Logistic L1,L2,Y1” and press ENTER. L1 is \( \text{2nd} \rightarrow \text{L1} \) and L2 is \( \text{2nd} \rightarrow \text{L2} \), while the Y1 can be found under \text{VARS} \rightarrow \text{Y-VARS} \rightarrow \text{1:Function} \rightarrow \text{Y1}. This command tells the calculator to perform a logistic regression with L1 on the horizontal axis, L2 on the vertical axis, and then put the resulting equation into Y1 (so you don’t have to copy it down and type it in yourself). See the SAMPLE SIMULATIONS at the end of this activity to see what you and the students should get as a result.
(11) After the students have created their table, have them work through the worksheet.
### SAMPLE SIMULATION #1

<table>
<thead>
<tr>
<th>Day</th>
<th>People selected randomly (that weren’t already infected)</th>
<th>Total number of people infected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 selected, 1 new (original host) 30</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1 selected, 1 new 22</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2 selected, 2 new 10,8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4 selected, 4 new 29,28,2,4</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>8 selected, 5 new 21,19,3,6,12</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>13 selected, 4 new 26,18,1,27</td>
<td>17</td>
</tr>
<tr>
<td>6</td>
<td>17 selected, 6 new 14,17,20,24,11,7</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>23 selected, 5 new 5,9,13,15,25</td>
<td>28</td>
</tr>
<tr>
<td>8</td>
<td>28 selected, 1 new 16</td>
<td>29</td>
</tr>
<tr>
<td>9</td>
<td>29 selected, 0 new</td>
<td>29</td>
</tr>
<tr>
<td>10</td>
<td>29 selected, 1 new 23</td>
<td>30</td>
</tr>
</tbody>
</table>

Here is a plot of the data points:

![Data Points Plot]

Note that the first four points follow the curve $y = 2^x$ exactly:

![Exponential Curve Plot]
But the actual curve is logistic in nature:

\[
y = \frac{c}{1 + ae^{(-bx)}}
\]

<table>
<thead>
<tr>
<th>Day</th>
<th>People selected randomly (that weren't already infected)</th>
<th>Total number of people infected</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 selected, 1 new (original host) 9</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1 selected, 1 new 25</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2 selected, 2 new 8,26</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4 selected, 3 new 22,1,11</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>7 selected, 5 new 7,5,30,4,3</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>12 selected, 8 new 16,2,29,28,19,17,20,24</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>20 selected, 4 new 23,13,27,6</td>
<td>24</td>
</tr>
<tr>
<td>7</td>
<td>24 selected, 3 new 21,10,8</td>
<td>27</td>
</tr>
<tr>
<td>8</td>
<td>27 selected, 2 new 12,14</td>
<td>29</td>
</tr>
<tr>
<td>9</td>
<td>29 selected, 0 new</td>
<td>29</td>
</tr>
<tr>
<td>10</td>
<td>29 selected, 1 new 15</td>
<td>30</td>
</tr>
</tbody>
</table>
Here is a plot of the data points:

Note that the first three points follow the curve $y = 2^x$ exactly:

But the actual curve is logistic in nature:

Logistic
\[ y = \frac{c}{1 + a e^{-bx}} \]

\[ a = 40.75212057 \]
\[ b = 0.8536930826 \]
\[ c = 30.00345081 \]
Logistic Functions
Worksheet A

(1) After simulating the first couple of days, what kind of curve did you expect to be able to use to model the given situation?

(2) What happened that made your suspected model inappropriate (i.e., did the number of infected people continue to grow exponentially)? Explain your answer by interpreting what you think was occurring regarding the spread of the disease.

A more appropriate model for what your data should show is called a logistic model. One way to write its equation is as follows: \( y = \frac{c}{1 + ae^{-bx}} \).

(3) Look at the results of your regression line and the values your calculator gave you for the constants \( a, b, \) and \( c \).
   a. What happens to the values of \( y \) as \( x \to \infty \)?

   b. What happens to the values of \( y \) as \( x \to 0^+ \)?

   c. Look at your answers for parts (a) and (b). These values should be familiar to you – they represent something about this situation – what do they represent? In light of this, what do you think (c) represents? Carry out another simulation on your own to confirm your thoughts.

   d. Again, look at your answers for parts (a) and (b). Why do these answers make sense? Explain your answer in terms of the spread of the disease and the number of people infected.
Recall that the average rate of change between two points can be found by finding the__________ of the line connecting those two points. The following questions focus on this idea.

a. What is the approximate average rate of change in the number of people infected during the first couple of days? Use units of measure (recall that \( m = \text{rise} / \text{run} \)) to explain what this average rate of change tells us.

b. Approximate the day on which the rate of change in the total number of people infected is the greatest.

c. Consider your answer to part (b). Compare the rate at which the population is being infected right before that day to the rate at which the population is being infected right after that day. What is happening to the rate of infection?

d. Can you explain what is happening to the population that would explain this change in the rate of infection?

FOR THE FUTURE: The day you found in question (4a) and the behavior you explained in question (4b) describes what is called an inflection point. Inflection points have numerous applications in physics, economics, statistics, and so on. We can determine how close your estimate was. You will not understand the math until calculus, but here is how you can determine the inflection point of your logistic curve:

a. Be sure that your logistic curve is still typed into Y1.

b. Type the following into Y2: nDeriv(nDeriv(Y1,X,X),X,X) and press the graph button. It will probably graph rather slowly, but be patient…a great discovery is at hand!!

c. Once the graph is finished, use you CALC – 2:zero function to find where the resulting graph (from part (b)) has an x-intercept. This x-value is the x-value at which the logistic curve has an inflection point, and should match your approximation from question (4b).
Logistic Functions
Worksheet A

(1) After simulating the first couple of days, what kind of curve did you expect to be able to use to model the given situation?

I expected to be able to use an exponential function.

(2) What happened that made your suspected model inappropriate (i.e., did the number of infected people continue to grow exponentially)? Explain your answer by interpreting what you think was occurring regarding the spread of the disease.

The data started to deviate a little bit at first, but eventually was very different from what an exponential model should have been.

A more appropriate model for what your data should show is called a logistic model. One way to write its equation is as follows: \[ y = \frac{c}{1 + ae^{-bx}}. \]

(3) Look at the results of your regression line and the values your calculator gave you for the constants \( a \), \( b \), and \( c \).

a. What happens to the values of \( y \) as \( x \to \infty \)?

The values of \( y \) appear to approach \( c \).

b. What happens to the values of \( y \) as \( x \to -\infty \)?

The values of \( y \) appear to approach 0. (Although time heading towards negative infinity doesn’t really make sense in the context of this problem, it does tell us something about the behavior of the graph overall.)

c. Look at your answers for parts (a) and (b). These values should be familiar to you – they represent something about this situation – what do they represent? In light of this, what do you think (c) represents? Carry out another simulation on your own to confirm your thoughts.

The values from part (a) and (b) represent the most and the least number of people that can be infected, respectively. I think \( c \) represents the actual total population.

d. Again, look at your answers for parts (a) and (b). Why do these answers make sense? Explain your answer in terms of the spread of the disease and the number of people infected.
My answers make sense because we can’t have less than 0 people infected, and we can’t have more than \( c \) people infected if there are only \( c \) people in the first place.

(4) Recall that the average rate of change between two points can be found by finding the __slope__ of the line connecting those two points. The following questions focus on this idea.

a. What is the approximate average rate of change in the number of people infected during the first couple of days? Use units of measure (recall that \( m = \frac{\text{rise}}{\text{run}} \)) to explain what this average rate of change tells us.

In simulation 1, the average rate of change is very small, around 2 people/day. This tells us the rate, in people per day, at which the population is becoming infected.

b. Approximate the day on which the rate of change in the total number of people infected is the greatest.

I would guess between day 4 and 5.

c. Consider your answer to part (b). Compare the rate at which the population is being infected right before that day to the rate at which the population is being infected right after that day. What is happening to the rate of infection?

Before that day, the rate at which the population is being infected is increasing and after that day, the rate is decreasing.

d. Can you explain what is happening to the population that would explain this change in the rate of infection?

The number of people that are infected is getting to the point where it is becoming more difficult to find someone who is not already infected, so the rate at which the population is becoming infected starts to decrease.

(5) **FOR THE FUTURE:** The day you found in question (4a) and the behavior you explained in question (4b) describes what is called an inflection point. Inflection points have numerous applications in physics, economics, statistics, and so on. We can determine how close your estimate was. You will not understand the math until calculus, but here is how you can determine the inflection point of your logistic curve:
a. Be sure that your logistic curve is still typed into Y1.
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c. Once the graph is finished, use you CALC – 2:zero function to find where the resulting graph (from part (b)) has an x-intercept. This x-value is the x-value at which the logistic curve has an inflection point, and should match your approximation from question (4b).

The actual inflection point seems to occur at day 4.496.

Topics for Further Exploration
(1) Have students go online and find 3 different applications of the logistic function.
(2) Have students explore the logistic map, i.e. \( f(x) = x(1-x) \), find fixed points, etc. Is there any relation to this and the logistic curve found above?
(3) Have students research the origin of the logistic function.
(4) Have the students each do a number (3 or 4) of simulations with 30 “pretend” students on their own, then compile the data and analyze the results.
   a. What seems to be the average x-value of the inflection point? Why would this make sense?
   b. On average, how many days does it take for the curve to diverge from an exponential?
   c. Use the data to determine experimental probabilities, such as “What is the probability that less than half of the population will be infected on day 6?” Then interpret the results, and determine if there is some way to figure out the probability theoretically.
   d. Have the students come up with ways that someone might use the data gathered in this simulation to help make well-informed decisions regarding prevention, treatment, etc. of the disease.