Subject §4.1

Let V and W be vector spaces, $L: V \to W$ be a mapping.

<u>Def</u>: $L: V \to W$ is a linear (mapping) operator if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

for any $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$. (two "·" two "+" in V, in W)

i.e., linear relation is preserved under a linear operator L.

Equivalent definition: $\begin{cases} L(x+y) = L(x) + L(y) \\ L(\alpha x) = \alpha L(x). \end{cases}$

<u>Remarks</u>:

1) $L(\theta_V) = \theta_W$: If $L(\theta_V) \neq \theta_W$, then L is not linear;

2) $L(-x) = -L(x)$;

3) $L(c_1 x_1 + \cdots + c_n x_n) = c_1 L(x_1) + \cdots + c_n L(x_n)$.

Linear operator from $\mathbb{R}^n$ to $\mathbb{R}^m$.

<u>THM</u>: An operator $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there is $A_{m\times n}$ such that $\quad L(x) = Ax \quad \forall x \in \mathbb{R}^n$.

<u>Proof</u>: ($\Leftarrow$) If $L(x) = Ax$, then $L(\alpha x_1 + \beta x_2) = A(\alpha x_1 + \beta x_2)$

$= \alpha A x_1 + \beta A x_2 = \alpha L(x_1) + \beta L(x_2)$.

So $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear.

($\Rightarrow$) Next assume that $L: \mathbb{R}^n \to \mathbb{R}^m$ is given linear, try to find $A_{m\times n}$ such that $L(x) = Ax$. $\forall x \in \mathbb{R}^n$.

Let $\{e_1, \cdots, e_n\}$ be the standard basis for $\mathbb{R}^n$,

denote
$$\begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix} = L(e_i), \quad i = 1, 2, \cdots, n.$$

set
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Let $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ be any given vector, we have

$$x = x_1 e_1 + \cdots + x_n e_n.$$

Since $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear, we have

$$L(x) = L(x_1 e_1 + \cdots + x_n e_n) = x_1 L(e_1) + \cdots + x_n L(e_n)$$
$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax;$$

linear combination of columns of A.

such $A_{m\times n}$ is called the representation matrix of $L: \mathbb{R}^n \to \mathbb{R}^m$.

<u>Ex.</u> a) $L(x) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$ is linear. Find RM A.

$L(e_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $L(e_2) = L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. $L(x) = Ax$

b) $L(x) = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$ is linear.

$L(e_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $L(e_2) = L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$L(x) = Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$.

Ex. Given any $x=\begin{bmatrix}x_0\\y_0\end{bmatrix}\in\mathbb{R}^2$, R_θ is the operator that rotates the vector x counterclockwise for an angle θ.

1) Show R_θ is linear.

2) Find A such that $R_\theta(x)=Ax$.

Do 2). Write $\begin{cases}x_0=r\cos\alpha\\y_0=r\sin\alpha\end{cases}$; $R_\theta(x)=R_\theta\begin{pmatrix}x_0\\y_0\end{pmatrix}=\begin{cases}r\cos(\alpha+\theta)\\r\sin(\alpha+\theta)\end{cases}$

$$=\begin{cases}r(\cos\alpha\cos\theta-\sin\alpha\sin\theta)=r\cos\alpha\cos\theta-r\sin\alpha\sin\theta\\r(\sin\alpha\cos\theta+\cos\alpha\sin\theta)=r\sin\alpha\cdot\cos\theta+r\cos\alpha\sin\theta\end{cases}$$

$$=\begin{cases}x_0\cos\theta-y_0\sin\theta\\y_0\cos\theta+x_0\sin\theta\end{cases}=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}x_0\\y_0\end{bmatrix}=Ax.$$

$A=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$. $x=\begin{bmatrix}x_0\\y_0\end{bmatrix}$. $R_\theta(x)=Ax$ linear

consider the physical meaning of R_θ.

we have $R_{-\theta}(x)=A^{-1}x=\begin{bmatrix}\cos(-\theta)&-\sin(-\theta)\\\sin(-\theta)&\cos(-\theta)\end{bmatrix}\begin{bmatrix}x_0\\y_0\end{bmatrix}$

$$A^{-1}=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}=A^T.$$

Ex. Given $L:\mathbb{R}^3\to\mathbb{R}^2$ linear such that

$L\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$, $L\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$, $L\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}$.

1) Find $L\begin{bmatrix}1\\2\\1\end{bmatrix}$.

2) Find $\begin{bmatrix}x\\y\\z\end{bmatrix}$, s.t. $L\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$.

3). Find A s.t. $Lx=Ax$.

Do 3) first. we have $L\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}=A\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}1&2&2\\2&1&3\end{bmatrix}$

$$\Rightarrow A=\begin{bmatrix}1&2&2\\2&1&3\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}^{-1}=\begin{bmatrix}1&2&2\\2&1&3\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&-1\\1&-1&0\end{bmatrix}=\begin{bmatrix}2&0&-1\\3&-2&1\end{bmatrix}$$

where

$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&1&0&0&1&0\\1&0&0&0&0&1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\1&1&0&0&1&0\\1&1&1&1&0&0\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&0&1&-1\\0&1&1&1&0&-1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&0&1&-1\\0&0&1&1&-1&0\end{array}\right]$$

Do 1) $L\begin{bmatrix}1\\2\\1\end{bmatrix}=A\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}2&0&-1\\3&-2&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$.

$x=t$, $~~z~~$ $z=2t-2$

Do 2). $L\begin{bmatrix}x\\y\\z\end{bmatrix}=A\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\end{bmatrix}$. $[A|\begin{smallmatrix}2\\6\end{smallmatrix}]=\left[\begin{array}{ccc|c}2&0&-1&2\\3&-2&1&6\end{array}\right]\to y=\dfrac{3t+(2t-2)-6}{2}$

$=\frac{5}{2}t-4$.

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\-4\\-2\end{bmatrix}+t\begin{bmatrix}1\\5/2\\2\end{bmatrix}.$$

Linear operator L from V to W

Ex: a) $L: C[a,b]\to\mathbb{R}$ defined by $L(f)=\int_a^b f(x)dx$ is linear.

(Integral operator is linear)

b) $D: C^1[a,b]\to C[a,b]$ defined by $D(f)=\dfrac{df(x)}{dx}$ is linear.

(Differential operator is linear)

c) $D^k: C^k[a,b]\to C[a,b]$ defined by $D^k(f)(x)=\dfrac{d^kf(x)}{dx^k}$

is linear.

Def: Let $L: V\to W$ be linear.

The kernel of L is defined by $\ker(L)=\{v\in V: L(x)=\theta_W\}$.

Let S be a subset of V, the image of S under L

is defined by $L(S)=\{w\in W: w=L(v), v\in S\}$.

The image of $L(V)$ is called the range of L denoted by $\mathcal{R}(L)$ = the set of all possible values that the operator L can take.

<u>THM</u>: If $L: V \to W$ is linear and S is a subspace of V, then

a) $\ker(L)$ is a subspace of V;

b) $L(S)$ is a subspace of W.

<u>Ex</u>: $L: \mathbb{R}^3 \to \mathbb{R}^2$. $x = (x_1, x_2, x_3)$. $L(x) = (x_1 + x_2, x_2 + x_3)^T$.

1) To find $\ker(L)$, solve for all x s.t. $L(x) = 0$

$\Leftrightarrow \begin{array}{l} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{array}$ set $x_2 = t$, $x_1 = -t$, $x_3 = -t$.

$\ker(L) = \text{span}\{(-1, 1, -1)\}$.

2). $L(x) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. $N(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$= \ker(L)$.

<u>Ex</u> Determine if the following operators are linear for each $p \in V = P_3 = \text{span}\{1, x, x^2\}$.

1) $L(p(x)) = x\,p(x)$ yes

2) $L(p(x)) = x^2 + p(x)$ No

3). $L(p(x)) = p(x) + x\,p(x) + x^2 p'(x)$ yes

1) $L(\alpha p_1(x)+\beta p_2(x)) = x(\alpha p_1(x)+\beta p_2(x))$

$= \alpha(x p_1(x)) + \beta(x p_2(x)) = \alpha L(p_1(x)) + \beta L(p_2(x))$. Yes

2) $L(\theta) = x^2 + \theta \neq \theta$ No.

3) $L(\alpha p_1(x)+\beta p_2(x)) = (\alpha p_1(x)+\beta p_2(x)) + x(\alpha p_1(x)+\beta p_2(x))$

$+ x^2(\alpha p_1'(x)+\beta p_2'(x)) = \alpha(p_1(x)+x p_1(x)+x^2 p_1'(x))$

$+ \beta(p_2(x)+x p_2(x)+x^2 p_2'(x)) = \alpha L(p_1(x)) + \beta L(p_2(x))$ yes.

Ex: Let $L_1: U \to V$, $L_2: V \to W$ be linear. Then

$L = L_2 \circ L_1 : U \to W$ defined by

$$L(u) = L_2(L_1(u))$$

is linear

Proof: $L(\alpha u_1 + \beta u_2) = L_2(L_1(\alpha u_1 + \beta u_2)) = L_2(\alpha L_1(u_1) + \beta L_1(u_2))$

$= \alpha L_2(L_1(u_1)) + \beta L_2(L_1(u_2)) = \alpha L(u_1) + \beta L(u_2)$. yes.

Ex: Find $\mathrm{Ker}(L)$ and $R(L)$, if $L: V \to W$. $V = P_3$

1) $L(p(x)) = x p'(x)$

2) $L(p(x)) = p(x) - p'(x)$

3) $L(p(x)) = p(0)x + p(1)$.

Let $p(x) = ax^2 + bx + c$ be any vector in P_3

1) $L(ax^2+bx+c)=x(2ax+b)=2ax^2+bx.$

$\mathcal{R}(L)=\text{span}\{x^2,x\}.$

For $Ker(L)$, solve $L(ax^2+bx+c)=2ax^2+bx=\theta$

$\Leftrightarrow a=b=0 \Rightarrow Ker(L)=\text{span}\{1\}.$

2) $L(ax^2+bx+c)=ax^2+bx+c-(2ax+b)=ax^2+(b-2a)x+c-b.$

$\mathcal{R}(L)=\text{span}\{x^2,x,1\}=P_3.$

For $Ker(L)$, solve $ax^2+(b-2a)x+c-b=0$

$\Rightarrow a=0 \Rightarrow b=0 \Rightarrow c=0.$ $Ker(L)=\{\theta\}.$

3) $L(ax^2+bx+c)=cx+a+b+c$

$\mathcal{R}(L)=\text{span}\{x,1\}$

For $Ker(L)$, solve $cx+a+b+c=0.$

$\Rightarrow c=0,\ a+b=0.$ $Ker(L)=\text{span}\{x^2-x\}$

ax^2-ax+0

Note $n=3=\dim(\mathcal{R}(L))+\dim(Ker(L))$

Ex: 1) $D: C^1[a,b]\to C[a,b]$. $D(f)(x)=\frac{df(x)}{dx}$. $Ker(D)=\{f=C \text{ on } [a,b]\}.$

2) $L: C^2[a,b]\to C[a,b]$ by $L(f)=a(x)f''(x)+b(x)f'(x)+c(x)f(x).$

$Ker(L)=\{f: a(x)f''(x)+b(x)f'(x)+c(x)f(x)=0.\ \text{diff. equation}\}.$