Subject §4.3 Similarity

Let $E=\{v_1,\cdots,v_n\}$ be a basis for V and $F=\{w_1,\cdots,w_m\}$ be a basis for W. $L:V\to W$ linear.

$$\begin{tikzcd} v\in V_E \arrow[r, "L"] \arrow[d, "E"'] & W_F \arrow[d, "F"] \\ {[v]_E}\in \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \ni [L(v)]_F = A[v]_E \end{tikzcd}$$

$$A=\left[[L(v_1)]_F\cdots[L(v_n)]_F\right].$$

Now let V be a vector space with $\dim(V)=n$, $L:V\to V$ be linear. The representation matrix of L depends on the basis we choose.

If E and F are two bases for V, A is representation matrix of L w.r.t. E, B is representation matrix of L w.r.t. F, i.e,

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "E"'] & V \arrow[d, "E"] \\ {[v]_E}\in\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n \ni [L(v)]_E = A[v]_E \end{tikzcd} ; \quad \begin{tikzcd} V \arrow[r, "L"] \arrow[d, "F"'] & V \arrow[d, "F"] \\ {[v]_F}\in\mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n \ni [L(v)]_F = B[v]_F \end{tikzcd}$$

$$A=\left[[L(v_1)]_E\cdots[L(v_n)]_E\right] \qquad B=\left[[L(w_1)]_F\cdots[L(w_n)]_F\right].$$

what is the relation between A and B?

Let T be the transition matrix from F to E

i.e., $E=\{v_1,\cdots,v_n\} \xleftarrow{T} F=\{w_1,\cdots,w_n\}$. express F in terms of E.

$T=\left[[w_1]_E \cdots [w_n]_E\right]$. Then we have the diagram

$$\begin{tikzcd}
V \arrow[r, "{[v]_E}"] \arrow[d, "L"'] & \mathbb{R}^n_E \arrow[d, "{A=\left[[L(w_1)]_E \cdots [L(w_n)]_E\right]}"] & & \mathbb{R}^n_F \arrow[ll, "T"'] \arrow[d, "{B=T^{-1}AT}"] \\
V \arrow[r, "{[L(v)]_E}"] & \mathbb{R}^n_E & & \mathbb{R}^n_F \arrow[ll, "T"']
\end{tikzcd}$$

Ex: $L:\mathbb{R}^2\to\mathbb{R}^2$ by $L\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2x_1\\x_1+x_2\end{bmatrix}$. Given two bases for $\mathbb{R}^2$,

$E=\{e_1,e_2\}$, $F=\{u_1,u_2\}$, $u_1=\begin{bmatrix}1\\1\end{bmatrix}$, $u_2=\begin{bmatrix}-1\\1\end{bmatrix}$.

1) The representation matrix A of L w.r.t. E is

$A=\left[[L(e_1)]_E [L(e_2)]_E\right]=\begin{bmatrix}2&0\\1+0&0+1\end{bmatrix}=\begin{bmatrix}2&0\\1&1\end{bmatrix}$

2) the representation matrix B of L w.r.t. F is

a) $B=\left[[L(u_1)]_F [L(u_2)]_F\right]=\left[\begin{bmatrix}2\\2\end{bmatrix}_F \begin{bmatrix}-2\\0\end{bmatrix}_F\right]=\begin{bmatrix}2&-1\\0&1\end{bmatrix}$.

$\begin{bmatrix}2\\2\end{bmatrix}=2\begin{bmatrix}1\\1\end{bmatrix}+0\begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}2\\2\end{bmatrix}_F=\begin{bmatrix}2\\0\end{bmatrix}$

$\begin{bmatrix}-2\\0\end{bmatrix}=-1\begin{bmatrix}1\\1\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix} \Rightarrow \begin{bmatrix}-2\\0\end{bmatrix}_F=\begin{bmatrix}-1\\1\end{bmatrix}$

b). $B=T^{-1}AT$. where $F\xrightarrow{T}E$. $T=\left[[u_1]_E [u_2]_E\right]=[u_1, u_2]=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$

$T^{-1}=\frac{1}{1+1}\begin{bmatrix}1&1\\-1&1\end{bmatrix}$

$B=T^{-1}AT=\frac{1}{2}\begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}2&0\\1&1\end{bmatrix}\begin{bmatrix}1&-1\\1&1\end{bmatrix}=\begin{bmatrix}2&-1\\0&1\end{bmatrix}$

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Def: Let A and B be $n \times n$ matrices. A and B are said to be similar if there is a nonsingular matrix U s.t.

$$B = U^{-1} A U. \quad \text{or} \quad A = U B U^{-1}.$$

denoted by $A \sim B$.

1) If A and B are representation matrices of a linear operator L w.r.t. two bases $E = \{v_1, \cdots, v_n\}$ and $F = \{u_1, \cdots, u_n\}$ respectively, then

$$A \sim B = T^{-1} A T$$

where $F \xrightarrow{T} E$. $T = \left[[u_1]_E \cdots [u_n]_E \right] = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \\ t_{n1} & \cdots & t_{nn} \end{bmatrix}$

$u_1 = t_{11} v_1 + \cdots + t_{n1} v_n$

$\cdots$

$u_n = t_{1n} v_1 + \cdots + t_{nn} v_n$.

Ex: $P_3 = \text{span}\{x^2, x, 1\}$, $D : P_3 \to P_3$ by $D(p(x)) = p'(x)$.

1) Find the matrix B representing D w.r.t. $\{1, x, x^2\} = E$

2) Find the matrix A representing D w.r.t. $\{1, 2x, 4x^2 - 2\} = F$

3) Find the matrix T such that $A = T^{-1} B T$.

1) $B = \left[[D(1)]_E \ [D(x)]_E \ [D(x^2)]_E \right] = \left[[0]_E \ [1]_E \ [2x]_E \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

2) $A = \left[[D(1)]_F \ [D(2x)]_F \ [D(4x^2-2)]_F \right] = \left[[0]_F \ [2]_F \ [8x]_F \right] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Since $0 = 0(1) + 0(2x) + 0(4x^2-2)$

$2 = 2(1) + 0(2x) + 0(4x^2-2)$

$8x = 0(1) + 4(2x) + 0(4x^2-2)$.

3) $F \xrightarrow{T} E$. $T = \left[[1]_E \ [2x]_E \ [4x^2-2]_E \right] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. To find T^{-1}.

$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right]$

$\Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$.

check

$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = T^{-1}BT = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ ✱.

Ex: $L: \mathbb{R}^3 \to \mathbb{R}^3$ has the representation matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ w.r.t. the basis $E = \{e_1, e_2, e_3\}$.

Find the matrix B representing L w.r.t. the basis $Y = \{y_1, y_2, y_3\}$ where $y_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $y_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

sol: Let $Y \xrightarrow{T} E$. Then $B = T^{-1}AT$. we have

$T = \left[[y_1]_E \ [y_2]_E \ [y_3]_E \right] = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & -1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1/3 & 1/3 & -2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right]$

$T^{-1} = \frac{1}{3} \begin{bmatrix} 0 & -3 & 3 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

Then $B = T^{-1}AT = \frac{1}{3}\begin{bmatrix}0&-3&3\\-1&-1&2\\1&1&1\end{bmatrix}\begin{bmatrix}2&2&0\\1&1&2\\1&1&2\end{bmatrix}\begin{bmatrix}1&-2&1\\-1&1&1\\0&1&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&1&0\\0&0&4\end{bmatrix}.$

Another way to find

$$B = \left[[Ay_1]_Y \ [Ay_2]_Y \ [Ay_3]_Y \right] = \left[\begin{bmatrix}0\\0\\0\end{bmatrix}_Y \begin{bmatrix}-2\\1\\1\end{bmatrix}_Y \begin{bmatrix}4\\4\\4\end{bmatrix}_Y \right] = \begin{bmatrix}0&0&0\\0&1&0\\0&0&4\end{bmatrix}$$

$$\begin{bmatrix}2&2&0\\1&1&2\\1&1&2\end{bmatrix}\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}^{y_1} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}2&2&0\\1&1&2\\1&1&2\end{bmatrix}\overbrace{\begin{bmatrix}-2\\1\\1\end{bmatrix}}^{y_2} = \begin{bmatrix}-2\\1\\1\end{bmatrix}, \quad \begin{bmatrix}2&2&0\\1&1&2\\1&1&2\end{bmatrix}\overbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}^{y_3} = \begin{bmatrix}4\\4\\4\end{bmatrix}. \quad ※$$

<u>Ex.</u> $P_3 = \text{span}\{x^2, x, 1\}$. $L: P_3 \to P_3$ by $L(p(x)) = xp'(x) + 2p(x)$

a) Find RM A of L w.r.t. $S = \{1, x, x^2\}$.

b) Find RM B of L w.r.t. $E = \{1, x, 1+x^2\}$.

c) Find T s.t. $B = T^{-1}AT$.

a) $A = \left[[L(1)]_S \ [L(x)]_S \ [L(x^2)]_S \right] = \left[[2]_S \ [x+2x]_S \ [2x^2+2x^2]_S \right] = \begin{bmatrix}2&0&0\\0&3&0\\0&0&4\end{bmatrix}$

b) $B = \left[[L(1)]_E \ [L(x)]_E \ [L(1+x^2)]_E \right] = \left[[2]_E \ [x+2x]_E \ [2x^2+2+2x^2]_E \right]$

$= \begin{bmatrix}2&0&-2\\0&3&0\\0&0&4\end{bmatrix}$, $\quad 4x^2+2 = -2+4(1+x^2)$

c) $E \xrightarrow{T} S$. $T = \left[[1]_S \ [x]_S \ [1+x^2]_S \right] = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

$T^{-1} = \begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$. check $B = T^{-1}AT$. ※