HWK — 11.3

Wednesday, April 17, 2019 2:33 PM

#1. Given $S: x^2+y^2+5z=1,\ z\geq 0$, upward normal

$F = xz\,\hat{i} + yz\,\hat{j} + (x^2+y^2)\,\hat{k}$. Verify Stokes Theorem.

$\Rightarrow S: z = g(x,y) = \frac{1}{5}(1-x^2-y^2) \geq 0 \Rightarrow D: x^2+y^2 \leq 1$

$N = (-g_x, -g_y, 1) = \left(\frac{2x}{5}, \frac{2y}{5}, 1\right)$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & x^2+y^2 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & yz \end{matrix} = (y, -x, 0)$$

$\nabla \times F \cdot N = \frac{2}{5}xy - \frac{2}{5}xy = 0$, $\Rightarrow LHS = 0$

For RHS. $\partial S: x^2+y^2=1,\ z=0 \Rightarrow X(s) = (\cos s, \sin s, 0)$

$X'(s) = (-\sin s, \cos s, 0)$

$F(X(s)) \cdot X'(s) = (0,0,1)\cdot(-\sin s, \cos s, 0) = 0 \Rightarrow RHS = 0$. ※

#4. Given $S.\ x^2+y^2+z^2=4,\ z\leq 0$, downward normal

$F = (2y-z)\,\hat{i} + (x+y^2-z)\,\hat{j} + (4y-3x)\,\hat{k}$. Verify Stokes theorem.

$\Rightarrow S: z = -\sqrt{4-x^2-y^2}$, $\partial S: z=0,\ x^2+y^2=4$, $X(s) = (2\cos s, 2\sin s, 0)$ $s\Big|_{2\pi}^{0}$

$N = (g_x, g_y, -1) = \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, -1\right)$,

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y-z & x+y^2-z & 4y-3x \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2y-z & x+y^2-z \end{matrix} = (5, 2, -1)$$

$$\nabla \times F \cdot N = \frac{\overbrace{(5x+2y)}^{\text{odd in } D}}{(4-x^2-y^2)^{1/2}} + 1$$

$$\iint_D \nabla\times F\cdot N\,dxdy = \iint_D 1\,dxdy = 4\pi. = LHS$$

For RHS, $x'(s) = (-2\sin s, 2\cos s, 0)$

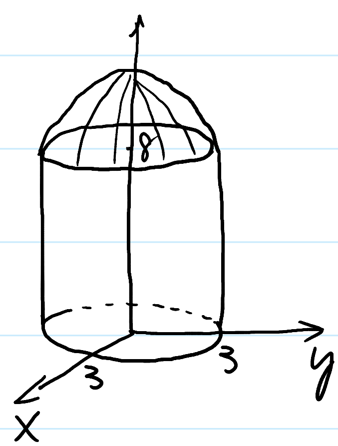
$$F(x(s))\cdot x'(s) = (4\sin s, 2\cos s + 4\sin^2 s, *)\cdot(-2\sin s, 2\cos s, 0)$$

$$= -8\sin^2 s + 4\cos^2 s + 8\sin^2 s\cdot\cos s$$

$$\int_{2\pi}^{0} F(x(s))\cdot x'(s)\,ds = \int_{2\pi}^{0}(-8\sin^2 s + 4\cos^2 s + \overset{=0}{8\sin^2 s\cos s})\,ds$$

$$= \int_{2\pi}^{0}(-8\cdot\tfrac{1}{2} + 4\tfrac{1}{2})\,ds = 4\pi = RHS.$$

$\sin^2 s = \frac{1}{2}(1-\cos 2s)$, $\cos^2 s = \frac{1}{2}(1+\cos 2s)$



#5. Given $S = S_1 \cup S_2$

$S_1: x^2+y^2 = 9, \quad 0\le z\le 8$

$S_2: x^2+y^2+(z-8)^2 = 9, \quad z\ge 8$

$F = (x^3+xz+yz^2)i + (xyz^3+y^7)j + x^2z^5k$,

Determine $\iint_S \nabla\times F\cdot d\vec{S} \ \left(= \oint_{\partial S} F\cdot d\vec{s} = I\right)$

$\partial S: x(t) = (3\cos t, 3\sin t, 0) \qquad 0\le t\le 2\pi$

$x'(t) = (-3\sin t, 3\cos t, 0)$.

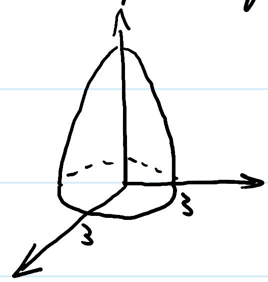
$$F(x(t))\cdot x'(t) = (3^3\cos^3 t, 3^7\sin^7 t, 0)\cdot(-3\sin t, 3\cos t, 0)$$

$$= -3^4\cos^3 t\sin t + 3^8\sin^7 t\cos t$$

$$I = \int_0^{2\pi} F(x(t))\cdot x'(t)\,dt = \left(\frac{3^4}{4}\cos^4 t + \frac{3^8}{8}\sin^8 t\right)\Big|_0^{2\pi} = 0$$

#6 Given $F=(x,y,z)$, $D=\{(x,y,z): 0\le z\le 9-x^2-y^2\}$

Verify divergence theorem.



$\nabla\cdot F = 3$.

$$\iiint_D \nabla\cdot F\,dV = 3\iint\left(\int_{z=0}^{z=9-x^2-y^2} dz\right)dx\,dy$$

$$= 3\iint (9-x^2-y^2)\,dx\,dy \qquad \begin{cases} x = r\cos\theta & 0\le\theta\le 2\pi \\ y = r\sin\theta & 0\le r\le 3 \end{cases}$$

$$= 3\int_0^{2\pi}\int_0^3 (9-r^2)\,r\,dr\,d\theta = 3\cdot 2\pi\left(\frac{9}{2}r^2-\frac{r^4}{4}\right)\Big|_0^3 = \frac{3}{2}81\pi.$$

For LHS, $S=S_1\sqcup S_2$. $S_1: z=9-x^2-y^2$, $S_2: \begin{cases} x = s\cos t \\ y = s\sin t \\ z = 0 \end{cases}$

$N_1 = (-g_x, -g_y, 1) = (2x, 2y, 1)$,

$$N_2 = T_s \times T_t = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = rk \to -rk \Rightarrow F\cdot N = 0 \text{ on } S_2.$$

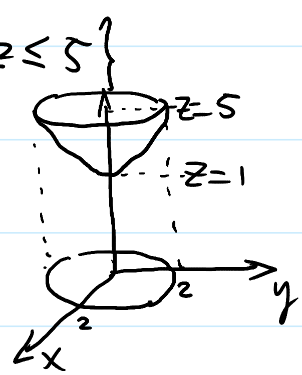
$$LHS = \iint_{S_1} (2x^2+2y^2+(9-x^2-y^2))\,dx\,dy = \iint_{S_1} (x^2+y^2+9)\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 (r^2+9)\,r\,dr\,d\theta \qquad \begin{cases} x = r\cos\theta, & 0\le\theta\le 2\pi \\ y = r\sin\theta, & 0\le r\le 3 \end{cases}$$

$$= 2\pi\left(\frac{r^4}{4}+\frac{9r^2}{2}\right)\Big|_0^3 = \frac{3}{2}81\pi = RHS.$$ #

#8: Given $F=(x^2,y,z)$, $D=\{(x,y,z): x^2+y^2+1\le z\le 5\}$

Verify Divergence Theorem



$$\oint\!\!\oint F\cdot d\vec{S} = \iint_{\partial D} F(X(s,t))\cdot N(s,t)\,ds\,dt$$

$$= \iiint_D \nabla\cdot F\,dV$$

$\nabla\cdot F = 2x+1+1 = 2x+2.$

$$\iiint_D (\overset{\to\text{odd}}{2x}+2)\,dV = 2\iiint_D dV = 2|D| = \iiint_{z=x^2+y^2+1}^{z=5} dz\,dx\,dy$$

$$=2\iint (4-x^2-y^2)\,dx\,dy = 2\int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta \quad \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

$0\le r\le 2,\ 0\le\theta\le 2\pi$

$$=16\pi$$

LHS: $S = S_1(\text{Top}) \cup S_2(\text{side})$: $S_1: \begin{cases} x=s\cos t \\ y=s\sin t \\ z=5 \end{cases}$ $\quad 0\le t\le 2\pi$, $\quad 0\le s\le 2$

$$N_1 = T_s\times T_t = \begin{vmatrix} i & j & k \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = sk = (0,0,s)$$

$$\iint_{S_1} F\cdot N\,ds\,dt = \int_0^2\int_0^{2\pi} 5s\,dt\,ds = 20\pi.$$

S_2: $z = x^2+y^2+1$, $N = (g_x, g_y, -1) = (2x, 2y, -1)$

$$F\cdot N = 2x^3+2y^2-(x^2+y^2+1) = \overset{\text{odd}}{2x^3} + y^2 - x^2 - 1. \quad \begin{matrix} x=r\cos\theta \\ y=r\sin\theta \\ 0\le r\le 2 \\ 0\le\theta\le 2\pi \end{matrix}$$

$$\iint_{S_2} F\cdot N\,ds\,dt = \int_0^{2\pi}\int_0^2 (2r^3\cos^3\theta + r^2\sin^2\theta - r^2\cos^2\theta - 1)\,r\,dr\,d\theta.$$

$$=\int_0^{2\pi}\Big(\underset{0}{\tfrac{2}{5}2^5\cos^3\theta} + \underbrace{\tfrac{2^4}{4}\sin^2\theta - \tfrac{2^4}{4}\cos^2\theta}_{4\left(\frac{1-\cos 2\theta}{2} - \frac{1+\cos 2\theta}{2}\right) = 0} - \tfrac{2^2}{2}\Big)\,d\theta = -4\pi$$

$=-4\cos(2\theta) \Rightarrow -2\sin(2\theta)\Big|_0^{2\pi} = 0.$

$$LHS = 20\pi - 4\pi = 16\pi = RHS.$$

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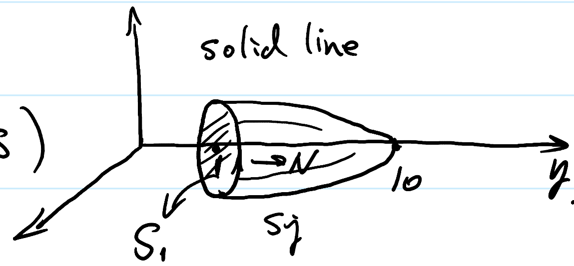
#11. Given $F = (2xyz + 5z,\ e^{x}\cos yz,\ x^2 y)$

$S: y = 10 - x^2 - z^2,\ y \geq 1.$

Determine $\iint_S \nabla \times F \cdot d\vec{S} \left(= \oint_{\partial S} F \cdot d\vec{s}\right)$



create $S_1: X(s,t) = (s\sin t,\ 1,\ s\cos t)$ with the same ∂S, $0 \leq s \leq 3,\ 3^2 = 10 - 1,\ t\big|_0^{2\pi}$

$$N = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \sin t & 0 & \cos t \\ s\cos t & 0 & -s\sin t \end{vmatrix} = s\,\hat{j}$$

only need to know $\hat{j}$ component

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz+5z & e^{x}\cos yz & x^2 y \end{vmatrix} = *\hat{i} + (2xy + 5 - 2xy)\hat{j} + *k = (*, 5, *)$$

$$\iint_{S_1} \nabla \times F \cdot d\vec{S} = \int_0^{2\pi}\int_0^3 (5s)\,ds\,dt = 5|D| = 5 \cdot 3^2\pi = 45\pi$$