M311-HWK-Test 2

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§4.1 #4 $L\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} = A\begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 3 & 2\end{bmatrix}$

$A = \begin{bmatrix}-2 & 5\\ 3 & 2\end{bmatrix}\frac{1}{-1-2}\begin{bmatrix}-1 & -1\\ -2 & 1\end{bmatrix} = \begin{bmatrix}8/3 & -7/3\\ 7/3 & 1/3\end{bmatrix}$,

$L\begin{bmatrix}7\\ 5\end{bmatrix} = A\begin{bmatrix}7\\ 5\end{bmatrix}$.

#19 Find the Kernel and Range of L on P_3

a) $L(p(x)) = xp'(x)$

(b) $L(p(x)) = p(x) - p'(x)$

(c) $L(p(x)) = p(0)x + p(1)$

Take $p(x) = ax^2 + bx + c$

(a) $L(p(x)) = x(2ax + b) = 2ax^2 + bx$.

$R(L) = \text{span}\{x, x^2\}$

set $L(p(x)) = 2ax^2 + bx = 0 \Rightarrow a = b = 0$.

$K(L) = \text{span}\{1\}$

(b) $L(p(x)) = ax^2 + bx + c - (2ax + b)$
$= ax^2 + (b-2a)x + c - b$

$R(L) = \text{span}\{1, x, x^2\}$

set $L(p(x)) = ax^2 + (b-2a)x + c - b = 0$

$\Rightarrow a = 0 \Rightarrow b = 0 \Rightarrow c = 0 \Rightarrow K(L) = \{0\}$

(c) $L(p(x)) = cx + a + b + c$

$\Rightarrow R(L) = \text{span}\{1, x\}$.

set $L(p(x)) = cx + a + b + c = 0 \Rightarrow c = 0,\ a+b=0$

$\Rightarrow ax^2 + bx + c = ax^2 - ax = a(x^2 - x), \Rightarrow K(L) = \text{span}\{x^2 - x\}$

§4.2 #13 $\quad L: P_2 \to \mathbb{R}^2$

$L(p(x)) = \begin{bmatrix} \int_0^1 p(x)dx \\ p(0) \end{bmatrix}$ Find matrix A s.t. $L(\alpha + \beta x) = A\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Use the bases $\{1, x\}$ for P_2 and $E = \{e_1, e_2\}$ for $\mathbb{R}^2$

$A = \left[[L(1)]_E \ [L(x)]_E \right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_E \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}_E \right] = \begin{bmatrix} 1 & 1/2 \\ 1 & 0 \end{bmatrix}$

$L(\alpha + \beta x) = \alpha L(1) + \beta L(x) = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

#14. $L(p(x)) = p'(x) + p(0)$. $L: P_3 \to P_2$. Find MR A w.r.t. bases $E = [x^2, x, 1]$ for P_3 and $F = [2, 1-x]$ for P_2

$A = \left[[L(x^2)]_F [L(x)]_F [L(1)]_F \right] = \left[[2x]_F [1]_F [1]_F \right] = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix}$

a) $[L(x^2 + 2x - 3)]_F = A\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -2 \end{bmatrix}$

b) $[L(x^2 + 1)]_F = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -2 \end{bmatrix}$

b) $[L(x^2+1)]_F = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -2 \end{bmatrix}$

c) $[L(3x)]_F = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix}$

d) $[L(4x^2+2x)]_F = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$.

#15 $S = \text{span}\{e^x, xe^x, x^2e^x\} \subset C[a,b]$.
$D(f)(x) = f'(x)$. Find MR A of D w.r.t. S.

$A = \left[[D(e^x)]_S \; [D(xe^x)]_S \; [D(x^2e^x)]_S \right]$
$= \left[[e^x]_S \; [e^x+xe^x]_S \; [2xe^x+x^2e^x]_S \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

§4.3 #2 $U = [u_1\, u_2]$. $V = [v_1, v_2]$ in $\mathbb{R}^2$
$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$L(x) = (-x_1, x_2)^T$.

a) Find transition matrix S from U to V,
b) Find matrix A of L w.r.t. V by computing SBS^{-1} where B is the matrix of L w.r.t. U.

a). $S = \left[[u_1]_V \; [u_2]_V \right] = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

b). $B = \left[[L(u_1)]_U \ [L(u_2)]_U \right] = \left[\begin{bmatrix}-1\\1\end{bmatrix}_U \begin{bmatrix}1\\1\end{bmatrix}_U \right] = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$$A = \begin{bmatrix}1&1\\-1&-3\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\-1&-3\end{bmatrix}^{-1} = \begin{bmatrix}1&1\\-3&-1\end{bmatrix}\frac{1}{-3+1}\begin{bmatrix}-3&-1\\1&1\end{bmatrix} = \frac{-1}{2}\begin{bmatrix}-2&0\\8&2\end{bmatrix}$$

c).

#5. $L: P_3 \to P_3$. $L(p(x)) = xp'(x) + p'(x)$

(a) Find MR A of L w.r.t $[1, x, x^2] = E$

(b) Find MR B of L w.r.t $[1, x, 1+x^2] = F$

(c) Find S s.t. $B = S^{-1}AS$.

(d) If $p(x) = a_0 + a_1x + a_2(1+x^2)$, calculate $L^n(p(x))$.

a) $A = \left[[L(1)]_E \ [L(x)]_E \ [L(x^2)]_E \right] = \left[(0)_E \ [x]_E \ [2x^2+2]_E \right] = \begin{bmatrix}0&0&2\\0&1&0\\0&0&2\end{bmatrix}$

b) $B = \left[[L(1)]_F \ [L(x)]_F \ [L(1+x^2)]_F \right] = \left[[0]_F \ [x]_F \ [2x^2+2]_F \right] = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}$

c) $S: F \to E$: $S = \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$

$$\begin{tikzcd} V \arrow[r] \arrow[d, "L"'] & \mathbb{R}^3_E \arrow[d, "A"] & \mathbb{R}^3_F \arrow[l, "S"'] \arrow[d, "B"] \\ V & \mathbb{R}^3_E & \mathbb{R}^3_F \arrow[l, "S"'] \end{tikzcd} \quad B = S^{-1}AS$$

d) $[L(p(x))]_F = B[p(x)]_F = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$, $B^n = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2^n\end{bmatrix}$

$$[L^n(p(x))]_F = B[L^{n-1}(p(x))]_F \cdots = B^n[p(x)]_F = \begin{bmatrix}0&0&0\\0&1&0\\0&0&2^n\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}$$

$= \begin{bmatrix}0\\a_1\end{bmatrix}$ $\Rightarrow L^n(p(x)) = 0\cdot 1 + a_1\cdot x + 2^n a_2(1+x^2)$

$$= \begin{bmatrix} 0 \\ a_1 \\ 2^n a_2 \end{bmatrix}. \Rightarrow L^n(p(x)) = 0\cdot 1 + a_1 \cdot x + 2^n a_2 (1+x^2)$$

§5.1 #3(d), $x = (2,-5,4)^T$, $y = (1,2,-1)^T$.

$$p = x^T \cdot \frac{y}{\|y\|} \cdot \frac{y}{\|y\|} = \frac{-12}{6}(1,2,-1)^T = (-2,-4,2)^T.$$

Check $p \perp x-p$. $x-p = (+4,-1,2)^T$.

$p^T \cdot (x-p) = -8+4+4 = 0$ Yes.

#6. $y = 2x+1$, $P_0 = (5.2) \Rightarrow \bar{y} = y-1$.

then $\bar{y} = 2x$, $\bar{P}_0 = (5,1)$, $a = 2$

$$P_c = (5,1)\cdot(1,2)\,(1,2)/(1+2^2) \text{ +(0,1)}$$

$$= (7/5, 14/5). \text{ +(0,1)}$$

#7. $y = \frac{4}{3}x$, $P_0 = (1,2)$, $a = 4/3$.

$$d = (1,2)\cdot(1,4/3) / \sqrt{1+(4/3)^2}$$

$$= \frac{11}{3}\cdot\frac{3}{5} = 11/5$$

#8 (b). $N = (-3,6,2)^T$. $P_0 = (4,2,-5)$

$$-3(x-4)+6(y-2)+2(z+5)=0$$

#11 $(2,1,-2)$ to $6(x-1)+2(y-3)+3(z+4)=0$

$$d=(2-1,1-3,-2+4)\cdot(6,2,3)\Big/\sqrt{6^2+2^2+3^2}$$

$$=8/7$$

§5.2 #1(d). $A=\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\1&1&2&2\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\0&1&2&2\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\0&0&1&1\end{bmatrix}$

$$\to\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&1&1\\0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&0\end{bmatrix}\begin{matrix}\leftarrow\\\leftarrow\\\leftarrow\\ \\\end{matrix}$$

$$\uparrow\ \uparrow\ \uparrow$$

$$R(A^T)=\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$$

$$N(A)=\text{span}\{(0,0,-1,1)^T\}$$

$$R(A)=\text{span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\1\\2\end{bmatrix}\right\}$$

$$A^T=\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&1&1&2\\0&1&1&2\end{bmatrix}\to\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

$$N(A^T)=\text{span}\{(-1,-1,-1,1)^T\}$$

#3. (b) $A=\begin{bmatrix}1&2&1\\1&-1&2\end{bmatrix}\to\begin{bmatrix}1&2&1\\0&-3&1\end{bmatrix}\to\begin{bmatrix}1&5&0\\0&-3&1\end{bmatrix}$

#3. (b). $A=\begin{bmatrix}1&2&1\\1&-1&2\end{bmatrix}\to\begin{bmatrix}1&2&1\\0&-3&1\end{bmatrix}\to\begin{bmatrix}1&5&0\\0&-3&1\end{bmatrix}$

$x_2=t$. $x_3=3t$, $x_1=-5$. $S^{\perp}=\operatorname{span}\{(-5,1,3)^T\}$.

§5.3. #3 (b). $A=\begin{bmatrix}1&1&3\\-1&3&1\\1&2&4\end{bmatrix}$ $b=\begin{bmatrix}-2\\0\\8\end{bmatrix}$.

Step 1. $L=A^TA$.

Step 2 $d=A^Tb$.

Step 3 solve $Lx=d$ for all least square solutions x.

#5

x	-1	0	1	2
y	0	1	3	9

Find the best least square fit by a linear (quadratic) function.

a) linear $p=c_0+c_1x$

b) quadratic
$p=c_0+c_1x+c_2x^2$

a)
$$\begin{array}{c c} & \begin{array}{cc} x^0 & x^1 \end{array} \\ \begin{array}{c} x_1\\x_2\\x_3\\x_4\end{array} & \begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix} \end{array} \qquad \begin{bmatrix}0\\1\\3\\9\end{bmatrix}$$
A Y

Solve $A^TA\,C=A^TY$ for $C=\begin{bmatrix}c_0\\c_1\end{bmatrix}$

b)
$$\begin{array}{c c} & \begin{array}{ccc} x^0 & x^1 & x^2 \end{array} \\ \begin{array}{c} x_1\\x_2\\x_3\\x_4\end{array} & \begin{bmatrix}1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{bmatrix} \end{array} \qquad \begin{bmatrix}0\\1\\3\\9\end{bmatrix}$$
A Y

$\begin{bmatrix}c_0\end{bmatrix}$

$$A \qquad Y$$

solve $A^T A C = A^T Y$ for $C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$.

§5.4 #1. $x = (-1,-1,1,1)^T$, $y = (1,1,5,-3)^T$.

show $x \perp y$, calculate $\|x\|_2, \|y\|_2, \|x+y\|_2$

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

#15. Compute $\|x\|_1, \|x\|_2, \|x\|_\infty$

a) $x = (-3, 4, 0)$.

$\|x\|_1 = 7$, $\|x\|_2 = 5$, $\|x\|_\infty = 4$,

§5.5

#3. Let u_2, u_3 be $\perp$-normal in $\mathbb{R}^3$ and $S = \text{span}\{u_2, u_3\}$.

Let $x = (1,2,2)^T$. Find the projection p of x onto S.

$$p = x^T \cdot u_2 \, u_2 + x^T \cdot u_3 \, u_3 \qquad u_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \; u_3 = \begin{bmatrix} 1/\sqrt{2} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \left(\tfrac{2}{3} + \tfrac{4}{3} + \tfrac{2}{3}\right) u_2 + \left(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}}\right) u_3$$

$$= \begin{bmatrix} 16/9 \\ 16/9 \\ 8/9 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} +23/18 \\ 41/18 \\ 8/9 \end{bmatrix}$$

$$x - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 23/18 \\ 41/18 \\ 8/9 \end{bmatrix} = \begin{bmatrix} -5/18 \\ -5/18 \\ 10/9 \end{bmatrix} \perp \underset{u_2}{\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}}, \underset{u_3}{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}$$

$\lfloor 2/3 \rfloor \quad \lfloor 8/9 \rfloor \quad \lfloor 10/9 \rfloor \qquad \lfloor 1/3 \rfloor \quad \lfloor 0 \rfloor$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_2 \qquad u_3$

#7. $\{u_1, u_2, u_3\}$ $\perp$-normal basis for an IPS V

$x = c_1 u_1 + c_2 u_2 + c_3 u_3$.

$\|x\| = 5$, $\langle u_1, x\rangle = 4$, $x \perp u_2$.

What are the possible values for c_1, c_2, c_3.

$c_1 = \langle x, u_1\rangle = 4$, $\quad c_2 = \langle x, u_2\rangle = 0$.

$5^2 = \|x\|^2 = c_1^2 + c_2^2 + c_3^2 = 4^2 + 0^2 + c_3^2$

$\Rightarrow c_3 = \pm 3$.

#8, #9. See HWK hints.

§5.6 #3, #4. See lecture notes 5.6+.

§6.1. #1 (f, g, h, l)

f). $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0$

$\lambda = 0$ triple root

$\lambda = 0$ triple root

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_3 = 0,\ x_2 = 0,\quad x_1 = t. \qquad V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

g) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0$$

$\lambda_1 = 1$ double, $\lambda_2 = 2$.

$$\lambda = 1.\ A - \lambda I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 = t,\ x_2 = -t,\ x_1 = s$$

$$V_1 = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.\quad V_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\quad V_1^2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2 \quad A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_2 = t,\ x_1 = t,\ x_3 = 0$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

h) $$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix},\ |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(\lambda-3)(\lambda+1) - 5] = (1-\lambda)(\lambda^2 - 2\lambda - 8)$$

$$= (1-\lambda)(\lambda - 4)(\lambda + 2) = 0$$

$\lambda_1 = -2,\ \lambda_2 = 1,\ \lambda_3 = 4.$

$$\lambda_1 = -2,\ A - \lambda_1 I = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -3 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = t,\ x_3 = -5t,\ x_1 = t.\quad V_1 = (1, 1, -5)^T.$

$$\lambda_2 = 1,\ A - \lambda_2 I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_2 = 0,\ x_3 = 0$$

$\lambda_2 = 1.\quad A - \lambda_2 I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 1 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = 0, x_3 = 0 \\ x_1 = t, \end{matrix}$

$V_2 = (1, 0, 0)^T.$

$\lambda_3 = 4,\quad A - \lambda_3 I = \begin{bmatrix} -3 & \overset{2}{\cancel{3}} & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{bmatrix} \to \begin{bmatrix} -3 & \overset{2}{\cancel{3}} & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & \overset{3}{\cancel{4}} & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_2 = t,\ x_3 = t,\ x_1 = \frac{\cancel{4}\,1}{3} t.\quad V_3 = (\cancel{\tfrac{4}{3}}\ 1, 1, 1)^T$ or $\cancel{(4, 3, 3)}^T.$

l) $A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},\ |A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 & 0 \\ 4 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$

$= (3-\lambda) \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(2-\lambda)^2 = 0$

$\lambda_1 = 1.\ \lambda_2 = 2$ double, $\lambda_3 = 3.$

$\lambda_1 = 1.\quad A - \lambda_1 I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} x_4 = 0, x_3 = 0, x_1 = 0 \\ x_2 = t. \end{matrix}$

$V_1 = (0, 1, 0, 0)^T.$

$\lambda_2 = 2.\quad A - \lambda_2 I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} x_4 = 0, x_1 = 0.\ x_2 = 0 \\ x_3 = t. \end{matrix}$

$V_2 = (0, 0, 1, 0)^T.$

$\lambda_3 = 3$. $A - \lambda_3 I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{matrix} x_4 = 0, x_3 = 0 \\ x_1 = t. \ x_2 = 2t. \end{matrix}$

$V_3 = (1, 2, 0, 0)^T$.

§6.2. #1 (b, d, f). #2 (a, c)

#1 (b). $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{vmatrix} = (\lambda-2)(\lambda+3)+4$

$= \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) = 0$

$\lambda_1 = -2$, $\lambda_2 = 1$.

$\lambda_1 = -2$, $A - \lambda_1 I = \begin{bmatrix} 4 & 4 \\ -1 & -1 \end{bmatrix}$ $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1$. $A - \lambda_2 I = \begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix}$. $V_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

the general solution

$$Y(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + 4 c_2 e^{t} \\ -c_1 e^{-2t} - c_2 e^{t} \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

#1 (d) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1$

$= \lambda^2 - 2\lambda + 2 = 0$.

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm i$$

$\lambda_1 = 1 + i$. $A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ $\begin{matrix} x_1 = t \\ x_2 = -it \end{matrix}$ $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_1 = 1+i$. $A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$ $\begin{matrix} x_1 = t \\ x_2 = -it \end{matrix}$ $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

the general solution is

$$Y(t) = e^t \left[C_1 \left(\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + C_2 \left(\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right]$$

$$= \begin{bmatrix} e^t (C_1 \cos t + C_2 \sin t) \\ e^t (-C_1 \sin t + C_2 \cos t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

#1 (f) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix}$, $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 6 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{vmatrix}$

$$= (1-\lambda)\big((2-\lambda)(3-\lambda) - 6\big) = (1-\lambda)(\lambda^2 - 5\lambda)$$

$$= \lambda(1-\lambda)(\lambda - 5) = 0.$$

$\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 5$.

$\lambda_1 = 0$, $A - \lambda_1 I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_3 = t. \; x_2 = -3t \\ x_1 = -t. \end{matrix}$

$V_1 = (-1, -3, 1)^T$.

$\lambda_2 = 1$. $A - \lambda_2 I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{matrix} x_3 = x_2 = 0. \\ x_1 = t. \end{matrix}$

$V_2 = (1, 0, 0)^T$

$\lambda_3 = 5$. $A - \lambda_3 I = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} -4 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{matrix} x_1 = t. \; x_3 = 4t. \\ x_2 = 8t. \end{matrix}$

$V_3 = (1, 8, 4)^T$.

The general solution is

$$Y(t) = C_1 \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_3 e^{5t} \begin{bmatrix} 1 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -C_1 + C_2 e^{t} + C_3 e^{5t} \\ -3C_1 + 8C_3 e^{5t} \\ C_1 + 4C_3 e^{5t} \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

#2 (a, c)

(a) $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ I.C. $Y(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 1 - 4 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0.$$

$\lambda_1 = -3$, $\lambda_2 = 1$.

$\lambda_1 = -3$. $A - \lambda_1 I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1$, $A - \lambda_2 I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The general solution is

$$Y(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 e^{-3t} + C_2 e^{t} \\ -C_1 e^{-3t} + C_2 e^{t} \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

To determine C_1, C_2 by I.C.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = Y(0) = \begin{bmatrix} C_1 + C_2 \\ -C_1 + C_2 \end{bmatrix} \Rightarrow \begin{matrix} C_2 = 2 \\ C_1 = 1. \end{matrix}$$

The solution is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2e^{t} \\ -e^{-3t} + 2e^{t} \end{bmatrix}.$$

(c). $A = \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$, I.C $Y(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix} \begin{matrix} 2-\lambda & 0 \\ 1 & -\lambda \\ 0 & 1 \end{matrix}$$

$= (2-\lambda)\lambda(2+\lambda) - 6 + 3(2-\lambda)$

$= (2-\lambda)\lambda(2+\lambda) - 3\lambda = \lambda(4 - \lambda^2 - 3)$

$= \lambda(1-\lambda^2) = \lambda(1-\lambda)(1+\lambda) = 0.$

$\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$.

$\lambda_1 = -1$. $A - \lambda_1 I = \begin{bmatrix} 3 & 0 & -6 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 \\ 3 & 0 & -6 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $X_3 = t$. $X_2 = t$, $X_1 = 2t$ $\quad V_1 = (2, 1, 1)^T$.

$\lambda_2 = 0$. $A - \lambda_2 I = \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $X_3 = t$, $X_2 = 2t$, $X_1 = 3t$.

$V_2 = (3, 2, 1)^T$.

$\lambda_3 = 1$, $A - \lambda_3 I = \begin{bmatrix} 1 & 0 & -6 \\ 1 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$x_3 = t,\ x_2 = 3t,\ x_1 = 6t.\ V_3 = (6, 3, 1)^T.$

The general solution is

$$Y(t) = C_1 e^{-t}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + C_3 e^{t}\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2C_1 e^{-t} + 3C_2 + 6C_3 e^{t} \\ C_1 e^{-t} + 2C_2 + 3C_3 e^{t} \\ C_1 e^{-t} + C_2 + C_3 e^{t} \end{bmatrix} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}.$$

To determine C_1, C_2, C_3 by I.C.

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = Y(0) = \begin{bmatrix} 2C_1 + 3C_2 + 6C_3 \\ C_1 + 2C_2 + 3C_3 \\ C_1 + C_2 + C_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 6 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 3 & 6 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & -2 & 0 \end{array}\right]$$

$C_2 = -2 \Rightarrow C_3 = -1. \Rightarrow C_1 = 3$

the final solution is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 6e^{-t} - 6 - 6e^{t} \\ 3e^{-t} - 4 - 3e^{t} \\ 3e^{-t} - 2 - e^{t} \end{bmatrix}.$$