Math 311-Zhou **Review for Exam II** **2019**

1. Linear Transformation: $L : V \to W$, $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

2. $L : \mathbb{R}^n \to \mathbb{R}^m$ linear $\Leftrightarrow$ $L(x) = Ax$, where $A_{m\times n} = \Big[L(e_1)L(e_2)...L(e_n)\Big]$.

3. $L : V \to W$ linear. Find $\ker(L) = \{v \in V : L(v) = 0\}$ and $\mathfrak{R}(L) = \{L(v) : v \in V\}$. $n = r + k$ where $n = \dim(V)$, $r = \dim(\mathfrak{R}(L))$, $k = \dim(\ker(L))$.

4. Matrix Representation. $L : V \to W$ linear. $E = \{v_1, ..., v_n\}, F = \{u_1, ..., u_m\}$ bases for V and W. $[L(v)]_F = A[v]_E$ where $A = \Big[[L(v_1)]_F ...[L(v_n)]_F\Big]$. Pay attention to $L : P_n \to P_m$.

5. $L : V \to V$ linear. $E = \{v_1, ..., v_n\}, F = \{u_1, ..., u_n\}$ bases for V. Then $[L(v)]_E = A[v]_E$, $[L(v)]_F = B[v]_F$ where $A = \Big[[L(v_1)]_E \cdots [L(v_n)]_E\Big]$, $B = \Big[[L(u_1)]_F \cdots [L(u_n)]_F\Big]$.
Similarity. $B = T^{-1}AT$ and $T = \Big[[u_1]_E \cdots [u_n]_E\Big]$ is the transition matrix from $F \to E$.

6. In $\mathbb{R}^n$, scalar product $x^T y = \sum_{i=1}^n x_i y_i$. $x \perp y \Leftrightarrow x^T y = 0$. Given $S = \text{span}\{v_1, .., v_n\}$, find $S^\perp = \mathcal{N}(A)$ where $A = [v_1...v_n]^T$.

7. Inner product in $\mathbb{R}^n, C[a, b]$. $\langle x, y\rangle = \sum_{i=1}^{n} w_i x_i y_i$, $\langle f, g\rangle = \int_a^b w(x)f(x)g(x)dx$. $u \perp v \Leftrightarrow \langle u, v\rangle = 0$.
Normed Space. C-S inequality: $|\langle u, v\rangle| \le \|u\|\|v\|$, $\cos\theta = \frac{\langle u,v\rangle}{\|u\|\|v\|}$, $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p, \|\cdot\|_\infty$.

8. $V = S \oplus S^\perp$. For $A_{m\times n}$, $\mathcal{N}(A) = \mathfrak{R}(A^T)^\perp, \mathcal{N}(A^T) = \mathfrak{R}(A)^\perp$. $\mathbb{R}^n = \mathcal{N}(A) \oplus \mathfrak{R}(A^T)$. n=r+k.

9. Scalar proj. of u onto v: $\alpha = \frac{\langle u,v\rangle}{\|v\|}$ and vector proj. of u onto v: $p = \frac{\langle u,v\rangle}{\|v\|}\frac{v}{\|v\|}$. $\|v\|^2 = \langle v, v\rangle$.
Distance from a point $(x_p, y_p)/(x_p, y_p, z_p)$ to a line/plane $(y = ax)/(ax + by + cz = 0)$.
$d = \|(x_p, y_p) - P\|$ where $P = \frac{\langle (x_p,y_p),(1,a)\rangle (1,a)}{\|(1,a)\|^2}$ / $d = \frac{\langle (x_p,y_p,z_p),(a,b,c)\rangle}{\|(a,b,c)\|}$. Note
$\text{dis}((x_p, y_p, z_p), a(x-x_0)+b(y-y_0)+c(z-z_0)=0) = \text{dis}((x_p-x_0, y_p-y_0, z_p-z_0), ax+by+cz=0)$.

10. Projection onto a Subspace and Best Approximation. $S = \text{span}\{u_1, ..., u_n\} \subset V$ and $v \in V$. The proj. p of v onto $S \Leftrightarrow p - v \perp S \Leftrightarrow p - v \perp u_i, i = 1, ..., n$. Normal equation $AC = D$ where $A = (\langle u_i, u_j\rangle)_{(1\le i,j\le n)}$, $D = (\langle v, u_1\rangle, ..., \langle v, u_n\rangle)^T$, $C = (c_1, ..., c_n)^T$, $p = c_1u_1 + ... + c_nu_n$.
For $S = \text{span}\{u_1, ..., u_n\}$-an orthonormal basis, $p = \sum_{i=1}^{n} \langle v, u_i\rangle u_i$; $\langle u, v\rangle = \sum_{i=1}^{n} \langle u, u_i\rangle\langle v, u_i\rangle$.

11. When $\{u_1, ..., u_n\}$ is orthonormal, $v = c_1u_1 + ... + c_nu_n \Leftrightarrow c_1 = \langle v, u_1\rangle, ..., c_n = \langle v, u_n\rangle$.

12. Least squares solution x to $Ax = b$. Solve the normal equation $A^T Ax = A^T b$ for x. Then $p = Ax$ is the proj. of b onto $\mathfrak{R}(A)$.

13. Curve fitting: Given $\begin{array}{|c|c|c|c|} x & x_1 & \cdots & x_m \\ \hline y & y_1 & \cdots & y_m \end{array}$. Find least squares polynomial of degree $\leq n$. $p(x) = c_0 + c_1 x + \cdots + c_n x^n$ s.t. $p(x_i) = y_i$ (usually over-determined). Set

$$A_{m\times n} = \begin{bmatrix} x_1^0 & x_1^1 & \cdots & x_1^n \\ x_2^0 & x_2^1 & \cdots & x_2^n \\ . & . & \cdots & \\ x_m^0 & x_m^1 & \cdots & x_m^n \end{bmatrix}, C_{n\times 1} = \begin{bmatrix} c_0 \\ c_1 \\ \cdots \\ c_n \end{bmatrix}, D_{m\times 1} = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{bmatrix}.$$

Solve normal equation $A^T AC = A^T D$.

14. Orthogonal matrix $U = [u_1 \cdots u_n]$ if $u_i^T u_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \Leftrightarrow U^T U = I.$

15. Gram-Schimidt Process. Given LI $\{x_1, ..., x_n\}$. Find orthonormal $\{u_1, ..., u_n\}$ s.t. $\text{Span}\{x_1, ..., x_k\} = \text{Span}\{u_1, ..., u_k\}, k = 1, ..., n$ where
$v_1 = x_1, u_1 = v_1/\|v_1\|, \cdots, v_k = x_k - \sum_{i=1}^{k-1} \langle x_k, u_i \rangle u_i, u_k = v_k/\|v_k\|, k = 2, ..., n.$

16. E-values and E-vectors of $A_{n\times n}$. $AX = \lambda X$ $(X \neq 0) \Leftrightarrow p(\lambda) \equiv |A - \lambda I| = 0 \Rightarrow$ Find E-values $\lambda_1, ..., \lambda_k$. For each λ_i, solve $(A - \lambda_i I)X = 0$ for E-vector $X_i \neq 0$, $i = 1, ..., k$.

17. If $A_{n\times n}$ is real, E-values $\lambda = a + ib, \bar{\lambda} = a - ib$ and E-vectors $X = X_R + iX_I, \bar{X} = X_R - iX_I$ appear in complex conjugate pairing.

18. $AX_i = \lambda_i X_i, i = 1, ..., n \Rightarrow$ (a) $\lambda_1 \times \cdots \times \lambda_n = \det(A)$ and (b) $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \text{trace}(A)$.

19. $AX = \lambda X \Rightarrow A^2 X = \lambda^2 X, ..., A^k X = \lambda^k X$, $A^{-1}X = \frac{1}{\lambda} X$, and $p(A)X = p(\lambda)X$ where $p(x) = a_m x^m + \cdots + a_1 x + a_0$.

20. System of 1st ODE. $Y' = AY$. I.C. $Y(0) = Y_0$. Solve $AX = \lambda X$ for real $\lambda_1, ..., \lambda_n$ and their E-vectors $X_1, ..., X_n$. The general solution is

$$Y(t) = c_1 e^{\lambda_1 t} X_1 + \cdots + c_n e^{\lambda_n t} X_n$$

and then use I.C. $Y(0) = Y_0$ to solve for $c_1, ..., c_n$.

21. If $A_{2\times 2}$ has E-value $\lambda = a + ib$ and E-vector $X = X_R + iX_I$, then the general solution is

$$Y(t) = c_1 e^{at}[\cos(bt)X_R - \sin(bt)X_I] + c_2 e^{at}[\cos(bt)X_I + \sin(bt)X_R]$$

and then use I.C. $Y(0) = Y_0$ to solve for c_1, c_2.