Subject Matrix form M311 - LN1.2

m×n system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (1)$$

Coefficient matrix of the system (matrix = a rectangular array of numbers)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Augmented matrix of the system

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \Longleftrightarrow (1)$$

In the last example

Coefficient Matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$, augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$.

3 ele operations

1) Interchange the order of two equations;
2) multiply both sides of an equation by a nonzero #;
3) Add a multiple of one equation to another equation.

3 ele row operation

1) Interchange two rows;
2) multiply a row by a nonzero #;
3) Add a multiply of one row to another row.

(1) $(i) \rightleftarrows (j)$; (2) $\alpha(i)$; (3) $\alpha(i) + (j)$.

# Gauss-Jordan Elimination Method

to solve a linear system, write out its augmented matrix, convert it into simpler equivalent one (triangular) by 3 ele row operations. Then solve it

Ex:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \xrightarrow[-2(1)+\to(3)]{-3(1)+\to(2)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & -1 & -1 & -2 \end{array}\right] \xrightarrow[(2)\rightleftarrows(3)]{\text{To avoid fraction}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -7 & -6 & -10 \end{array}\right] \xrightarrow{-7(2)+\to(3)} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right] \to x_3 = 4$$

$-x_2 - 4 = -2,\ x_2 = -2$

$x_1 + 2(-2) + 4 = 3,\ x_1 = 3$

$(x_1, x_2, x_3) = (3, -2, 4)$.

Ex: Solve the system

$$\begin{aligned} -x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \xrightarrow{(1)\rightleftarrows(2)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

$$\xrightarrow[-3(1)+\to(4)]{-2(1)+\to(3)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right] \xrightarrow[-2(2)+\to(4)]{2(2)+\to(3)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right]$$

$$\xrightarrow{-1(3)+\to(4)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \to x_4 = 2$$

$-3x_3 - 2(2) = -13,\ x_3 = 3$

$-x_2 - 3 + 2 = 0,\ x_2 = -1$

$x_1 + (-1) + 3 + 2 = 6,\ x_1 = 2.$

solution $(x_1, x_2, x_3, x_4) = (2, -1, 3, 2)$.

## §1.2 Echelon Form

<u>Def.</u> A matrix is said to be in row Echelon form if

1) rows of all zeros are put at the bottom of the matrix,
2) The 1st nonzero entry in a row is 1, called leading 1,
3). All entries in the lower-left and below of a leading 1 are zeros.

<u>Ex:</u> Echelon forms, why?

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

<u>Ex:</u> Not Echelon forms, why?

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

* Any matrix can be converted to an Echelon form by using 3 ele row operations.

* The method of using 3 ele row operations to convert a matrix into an Echelon form is called Gauss Elimination Method.

## Why Echelon form?

When an augmented matrix is converted to an Echelon form, it is easy to tell if the system has a solution (consistent) or not; when it has a solution, it is easy to find the solutions.

Ex:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \xrightarrow[\substack{2(1)+\to(3) \\ -1(1)+\to(5)}]{(1)+\to(2)} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow[\substack{-1(2)+\to(4) \\ -1(2)+\to(5)}]{-2(2)+\to(3)}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-1(3)+\to(5)]{-1(3)+\to(4)} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

no solution

$$0x_1+0x_2+0x_3+0x_4+0x_5 = \begin{cases} -4 \\ -3 \end{cases}$$

change the problem by changing RHS

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$x_1+x_2+x_3+x_4+x_5=1$, $x_1+x_2+x_3+x_4+3=1$, $x_1+x_2-6+3=1$. $x_1=t$

$\Rightarrow$ $x_3+x_4+2x_5=0$, $x_3+x_4+2(3)=0$, $x_3+x_4=-6$ $x_3=s$

$x_5=3$

$\Rightarrow x_2=4-t$, $x_4=-6-s$ $\Rightarrow (x_1,x_2,x_3,x_4,x_5)=(t,4-t,s,-6-s,3)$

for any $s, t$.

For an $m \times n$ system.

when $m > n$ (more equations than unknowns) over-determined. Usually (but not always) inconsistent;

when $m < n$ (more unknowns than equations) under-determined, usually (but not always) consistent with infinitely many solutions.

**Def:** Reduced row Echelon form, if a matrix is

1) a row Echelon form,
2) The 1st nonzero entry in each row is the only nonzero entry in its column.

**Ex.** Last time:

Echelon form $\quad$ reduce Echelon form

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

* The reduced row Echelon form of a matrix is unique. the process of using 3 ele row operations to convert a matrix into its reduced row Echelon form is called the Gauss–Jordan Reduction.

Ex: Use G-J reduction to solve

$-x_1+x_2-x_3+3x_4=1.$

$3x_1+x_2-x_3-x_4=2.$

$2x_1-x_2-2x_3-x_4=3.$

Augmented Matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 1 \\ 3 & 1 & -1 & -1 & 2 \\ 2 & -1 & -2 & -1 & 3 \end{array}\right] \xrightarrow[2(1)+(3)]{3(1)+(2)} \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 1 \\ 0 & 4 & -4 & 8 & 5 \\ 0 & 1 & -4 & 5 & 5 \end{array}\right] \xrightarrow{(2)\rightleftarrows(3)}$$

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -4 & 5 & 5 \\ 0 & 4 & -4 & 8 & 5 \end{array}\right] \xrightarrow{-4(2)+(3)} \left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -4 & 5 & 5 \\ 0 & 0 & 12 & -12 & -15 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} +1 & -1 & +1 & -3 & -1 \\ 0 & 1 & -4 & 5 & 5 \\ 0 & 0 & 1 & -1 & -5/4 \end{array}\right]$$

G-J E

GE

In Echelon form

$$\xrightarrow[-(3)+(1)]{4(3)+(2)} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1/4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -5/4 \end{array}\right] \xrightarrow{(2)+(1)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1/4 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -5/4 \end{array}\right]$$

In reduced Echelon form

$x_1 = \frac{1}{4}+t$

$x_2 = -t$

$x_3 = -\frac{5}{4}+t$

3 equations, 4 unknowns, Freedom $=4-3=1$. set $x_4=t$.

$(x_1,x_2,x_3,x_4)=(\frac{1}{4}+t,-t,-\frac{5}{4}+t,t)$ for any $t$. G-J R.

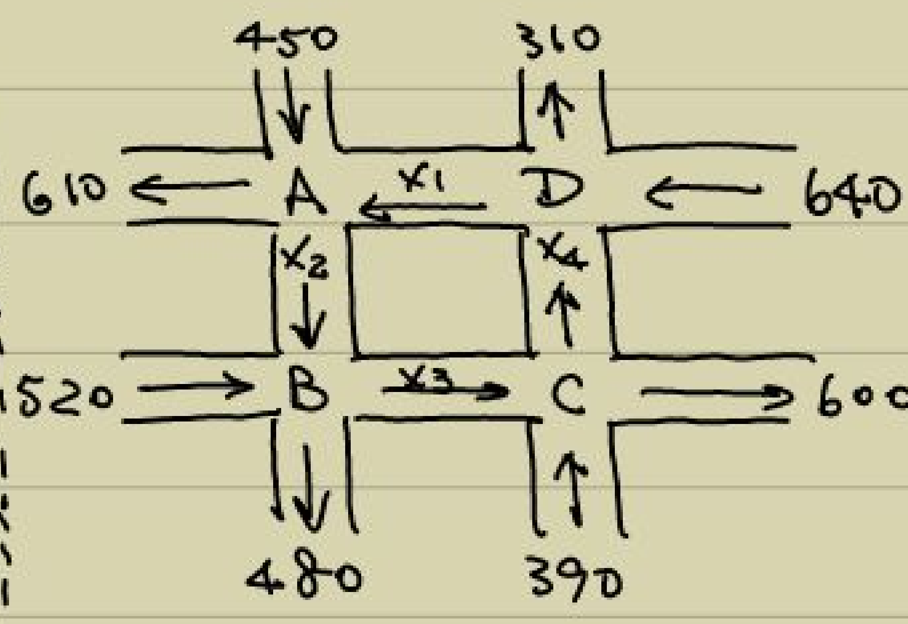
G-J E: solve the system;

G E: Echelon Form + solve the system;

G-J R: reduced Echelon form + solve the system.

An application. Traffic flow:

Two sets of one-way street intersections. Average hourly volume of traffic entering and leaving the streets.
To determine the average hourly volume of traffic $x_1, x_2, x_3, x_4$.




Intersection: Entering = leaving

A: $450 + x_1 = 610 + x_2$

B: $520 + x_2 = 480 + x_3$

C: $390 + x_3 = 600 + x_4$

D: $640 + x_4 = 310 + x_1$.

Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right]$$

$$\xrightarrow{(1)+\to(4)} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & -1 & 0 & 1 & -170 \end{array}\right] \xrightarrow{(2)+\to(4)} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & -1 & 1 & -210 \end{array}\right] \xrightarrow{(3)+\to(4)} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Freedom $= 4-3 = 1$. set $x_4 = s$. $x_3 = 210 + s$, $x_2 = -40 + 210 + s = 170 + s$

$x_1 = 160 + 170 + s = 330 + s$

reduced Echelon form

$$\xrightarrow{(3)+\to(2)} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(2)+\to(1)} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$