Subject §2.1 Determinants

Associated with each $n \times n$ matrix A, there is a number called determinant, denoted by $\det(A) = |A|$ such that $\det(A) \begin{cases} = 0 & \text{if } A \text{ is singular,} \\ \neq 0 & \text{if } A \text{ is nonsingular.} \end{cases}$

$n=1$. $A_{1\times 1} = a$, $\det(A) = a$.

$n=2$. $A_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

If $d = \det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, $A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

A is nonsingular.

$n=3$. $|A_{3\times 3}| = \left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right| = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{array}$

$- \quad - \quad - \quad + \quad + \quad +$

For $n > 3$, there is no such diagram for the determinant.

Def: Let $A = (a_{ij})_{n\times n}$. Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column containing a_{ij}. M_{ij} is called the minor of a_{ij} of A. The number

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

is called the cofactor of a_{ij}.

Cofactor expansion.

$$\det(A) = \sum_{k=1}^{n} a_{ik} A_{ik} = \sum_{k=1}^{n} a_{ik} (-1)^{i+k} \det(M_{ik})$$

along any row $i = 1, 2, \cdots, n$.

$$= \sum_{k=1}^{n} a_{kj} A_{kj} = \sum_{k=1}^{n} a_{kj} (-1)^{k+j} \det(M_{kj})$$

along any column $j = 1, 2, \cdots, n$.

<u>Ex.</u> $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$

1) By the diagram. $\det(A) = \left| \begin{matrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{matrix} \middle| \begin{matrix} 2 & 5 \\ 3 & 1 \\ 5 & 4 \end{matrix} \right| = \begin{matrix} 12+50+48 \\ -20-16-90 \end{matrix} = -16.$

2) By cofactor expansion

$M_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$, $M_{12} = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$, $M_{13} = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$,

$M_{21} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$. $M_{22} = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$, $M_{23} = \begin{bmatrix} 2 & 5 \\ 5 & 4 \end{bmatrix}$.

$M_{31} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $M_{32} = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $M_{33} = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$.

Along 1st row: $\det(A) = a_{11}(-1)^{1+1}|M_{11}| + a_{12}(-1)^{1+2}|M_{12}| + a_{13}(-1)^{1+3}|M_{13}|$

$= 2(1)(-2) + 5(-1)(+8) + 4(1)(7) = -16.$

Along 2nd row: $\det(A) = a_{21}(-1)^{2+1}|M_{21}| + a_{22}(-1)^{2+2}|M_{22}| + a_{23}(-1)^{2+3}|M_{23}|$

$= 3(-1)(14) + 1(1)(-8) + 2(-1)(-17) = -16.$

Along 3rd column: $\det(A) = a_{13}(-1)^{1+3}|M_{13}| + a_{23}(-1)^{2+3}|M_{23}| + a_{33}(-1)^{3+3}|M_{33}|$

$= 4(1)7 + 2(-1)(-17) + 6(1)(-13) = -16.$

Any row/column, choose a row/column containing most zeros.

Ex. Evaluate $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$ 1st column $= 2(-1)^{4+1}\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix}$ 3rd column $= -2\cdot 3(-1)^{3+3}(10-12)$

$= 12.$

§2.2.

THM. 1) $\det(A^T_{n\times n}) = \det(A_{n\times n})$;

2). If $A = (a_{ij})_{n\times n}$ is a triangular matrix, $|A| = a_{11}a_{22}\cdots a_{nn}$;

3). If A has a row/column of zeros, then $|A| = 0$;

4) If A has two identical rows/columns, then $|A| = 0$.

Determinants of $A_{n\times n}$ after 3 ele row operations:

If A_1 is obtained from A by

1) multiplying α to a row/column of A, then $|A_1| = \alpha|A|$;

2) adding a multiple of a row/column of A to another row/column.

then $|A_1| = |A|$;

3) interchanging two rows/columns of A, then $|A_1| = -|A|$.

Lemma. $A = (a_{ij})_{n\times n}$, A_{ij} = cofactor of a_{ij}. Then

$$a_{i1}A_{j1} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Rightarrow$ ith and jth rows are the same.

row operation II. $\alpha(i)$. $E_{II} = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & \alpha & & \\ & & & 1 & \\ 0 & & & & \ddots & 1 \end{bmatrix} \Rightarrow \det(E_{II}) = \alpha$.

$$\det(E_{II}A) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & & \cdots \\ \alpha a_{i1} & \cdots & \alpha a_{in} \\ \cdots & & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \alpha a_{i1}A_{i1} + \cdots + \alpha a_{in}A_{in}$$

$$= \alpha(a_{i1}A_{i1} + \cdots + a_{in}A_{in}) = \alpha \det(A)$$

$$= \det(E_{II})\det(A).$$

row operation III. $\alpha(i) \mapsto (j)$.

$E_{III} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \alpha & & \ddots & 1 \end{bmatrix}$, $\det(E_{III}) = 1$.

$$E_{III}A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & \cdots & \cdots & a_{in} \\ \alpha a_{i1} + a_{j1} & \cdots & \cdots & \alpha a_{in} + a_{jn} \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

do jth row cofactor expansion

$$\det(A) = (\alpha a_{i1} + a_{j1})A_{j1} + \cdots + (\alpha a_{in} + a_{jn})A_{jn}$$

$$= \alpha(\underbrace{a_{i1}A_{j1} + \cdots + a_{in}A_{jn}}_{i \neq j,\ 0}) + \underbrace{a_{j1}A_{j1} + \cdots + a_{jn}A_{n}}_{j=j.\ |A|.} = 0 + |A|.$$

$$= \det(A) = \det(E_{III})\det(A)$$

row operation I: $(i) \leftrightarrows (j) = -(j) \mapsto (i) + (i) \mapsto (j) + -(j) \mapsto (i) + -(i)$

(i)	$(i)-(j)$	$(i)-(j)$	$-(j)$	(j)
(j)	(j)	(i)	(i)	(i)

$$\begin{array}{c} a_{i1} \cdots a_{in} \\ \cdots A \\ a_{j1} \cdots a_{jn} \end{array} \to \begin{array}{c} a_{i1}-a_{j1} \cdots a_{in}-a_{jn} \\ A_1 \\ a_{j1} \cdots a_{jn} \end{array} \to \begin{array}{c} a_{i1}-a_{j1} \cdots a_{in}-a_{jn} \\ A_2 \\ a_{i1} \cdots a_{in} \end{array} \to \begin{array}{c} -a_{j1} \cdots -a_{jn} \\ A_3 \\ a_{i1} \cdots a_{in} \end{array}$$

$$\to \begin{array}{c} a_{j1} \cdots a_{jn} \\ A_4 \\ a_{i1} \cdots a_{in} \end{array} \qquad E_I = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots & 1 \end{bmatrix}, \ \det(E_I) = -1$$

$$\det(A_4) = -\det(A_3) = -\det(A_2) = -\det(A_1) = -\det(A)$$

$$\det(E_I A) = -\det(A) = \det(E_I)\cdot\det(A).$$

THM: If $E_1, E_2, \cdots, E_n$ are ele matrices, then

$$\det(E_n E_{n-1} \cdots E_2 E_1) = \det(E_n) \cdots \det(E_2) \cdot \det(E_1).$$

THM: An $n\times n$ matrix A is singular if and only if $\det(A) = 0$.

Proof: Use 3 ele row operations to convert A to its row reduced Echelon form U, i.e.,

$$E_n E_{n-1} \cdots E_2 E_1 A = U = \begin{cases} I & \text{if } A \text{ is nonsingular} \\ \text{contains a row of zeros} & \text{if } A \text{ is singular,} \end{cases}$$

$$\Rightarrow \det(U) = \begin{cases} 1 & \text{if } A \text{ is nonsingular}, \quad |E_i^{-1}| \neq 0. \\ 0 & \text{if } A \text{ is singular.} \end{cases}$$

$$|A| = \det(E_1^{-1}) \cdots \det(E_n^{-1}) \cdot \det(U) = \begin{cases} \neq 0 & \text{if } A \text{ nonsingular,} \\ 0 & \text{if } A \text{ singular.} \end{cases}$$

* How to use ele row operations to evaluate determinant.

Ex: $$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} \overset{\substack{-2(1)+to(2) \\ -3(1)+to(3)}}{=} \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = \begin{cases} 2\begin{vmatrix} 0 & -5 \\ -6 & -5 \end{vmatrix} = -60. \\ -\begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix} = -(2\cdot(-6)\cdot(-5)) = -60. \end{cases}$$

Ex: $$12 = \begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 1 & 3 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 4 & 5 & 0 \\ 0 & 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 5 & -12 \\ 0 & 0 & 3 & -6 \end{vmatrix}$$

$$\overset{-\frac{3}{5}(3)+to(4)}{=} \begin{vmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 5 & -12 \\ 0 & 0 & 0 & \frac{36}{5}-6 \end{vmatrix} = 2\cdot 1\cdot 5\cdot\left(\frac{36}{5}-6\right) = 2(36-30) = 12.$$

THM. Let A and B be $n\times n$ matrices. Then

$$\det(AB) = \det(A)\cdot\det(B).$$

$*\begin{cases} \text{If either } A \text{ or } B \text{ is singular, then so is } AB; \\ \text{If both } A \text{ and } B \text{ are nonsingular, then so is } AB. \end{cases}$

Proof: If A is singular, then $A = E_k \cdots E_1 U$ and U has a row of zeros. $\Rightarrow UB$ has a row of zeros

So $|AB| = |E_k \cdots E_1 U B| = |E_k| \cdots |E_1| \underbrace{|UB|}_{0} = 0.$

thus $$|AB| = |A||B| = 0$$

If A is nonsingular, then $A = E_k \cdots E_1 I$. $\Rightarrow |A| = |E_k| \cdots |E_1|$

$$|AB| = |E_k \cdots E_1 B| = |E_k| \cdots |E_1||B| = |A||B|.$$

$* \; |I| = 1, \quad A^{-1}A = I. \Rightarrow |A^{-1}| = \frac{1}{|A|}.$

Equivalent statements: $A_{n\times n}$

1) A is nonsingular,
2) $Ax = 0$ has only zero solution $x = 0$.
3) A is row equivalent to I.
4) $Ax = b$ has a unique solution for each b in $\mathbb{R}^n$;
5) $|A| \neq 0$.