Subject §3.3

<u>Def</u>: The vectors $v_1, \cdots, v_n$ in a vector space V are said to be linearly independent if

$$c_1 v_1 + \cdots + c_n v_n = \theta$$

holds only when $c_1 = c_2 = \cdots = c_n = 0$. Equivalently if no vector in $\{v_1, \cdots, v_n\}$ can be written as a linear combination of other vectors in $\{v_1, \cdots, v_n\}$.

$\{v_1, \cdots, v_n\}$ is said to be linearly dependent if there are scalars $c_1, \cdots, c_n$, not all zeros, such that

$$c_1 v_1 + \cdots + c_n v_n = \theta.$$

Equivalently, if one vector in $\{v_1, \cdots, v_n\}$ can be written as a linear combination of other vectors in $\{v_1, \cdots, v_n\}$.

<u>Ex</u>: In $\mathbb{R}^n$, since any vector x can be written as linear combination of $e_1, \cdots, e_n$, $\{e_1, \cdots, e_n, x\}$ is linearly dep.

In $\mathbb{R}^2$. Given v_1, if v_2 is linearly $\begin{matrix}\text{dep}\\ \text{indep}\end{matrix}$ to v_1 then span$\{v_1, v_2\}$ is $\begin{matrix}\text{a line}\\ \mathbb{R}^2\end{matrix}$.

In $\mathbb{R}^3$, Given v_1, v_2 linearly indep, span$\{v_1, v_2\}$ = a plane

v_3 is linearly $\begin{matrix}\text{dep}\\ \text{indep}\end{matrix}$ to $\{v_1, v_2\}$, span$\{v_1, v_2, v_3\}$ is $\begin{matrix}\text{a plane}\\ \mathbb{R}^3\end{matrix}$.

* Two vectors v_1 and v_2 are linearly dep $\Leftrightarrow \alpha v_1 = \beta v_2$.

<u>Ex.</u> Determine if the vectors are linearly indep.

a) $\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\}$,

b) $\{(1,0,1)^T, (0,1,0)^T\}$,

c) $\{(1,2,4)^T, (2,1,1)^T, (4,-1,1)^T\}$.

a). check the linear system has a nonzero solution or not.

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}c_1=0\\c_2=0\\c_3=0\end{matrix}$$, only zero solution. $\Rightarrow$ linearly indep

b) check the linear system $c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow c_1 = c_2 = 0$

only zero solution $\Rightarrow$ linearly indep.

c) check the linear system: $c_1\begin{bmatrix}1\\2\\4\end{bmatrix} + c_2\begin{bmatrix}2\\1\\3\end{bmatrix} + c_3\begin{bmatrix}4\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$,

or $\begin{bmatrix}1&2&4\\2&1&-1\\4&3&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has only zero solution if $\begin{vmatrix}1&2&4\\2&1&-1\\4&3&1\end{vmatrix} \neq 0$

we have $\begin{vmatrix}1&2&4\\2&1&-1\\4&3&1\end{vmatrix} = \begin{vmatrix}1&2&4\\0&-3&-9\\0&-5&-15\end{vmatrix} = \begin{vmatrix}1&2&4\\0&-3&-9\\0&0&0\end{vmatrix} = 0 \Rightarrow$ has nonzero solution for c_1, c_2, c_3

$\Rightarrow$ linearly dep.

THM: Let $A_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}, A_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix}, \cdots, A_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$ be vectors in $\mathbb{R}^n$

and $A = [A_1 A_2 \cdots A_n] = \begin{bmatrix} a_{11} a_{12} \cdots a_{1n} \\ a_{21} a_{22} \cdots a_{2n} \\ \vdots \\ a_{n1} a_{n2} \cdots a_{nn} \end{bmatrix}$.

Then $\{A_1, \cdots, A_n\}$ is $\begin{matrix} LD \\ LI \end{matrix}$ if and only if A is $\begin{matrix} \text{singular} \\ \text{nonsingular} \end{matrix}$.

Ex: $X_1 = (4, 2, 3)^T, X_2 = (2, 3, 1)^T, X_3 = (2, -5, 3)^T$.

$X = [X_1 X_2 X_3] = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{bmatrix}, |X| = \left|\begin{matrix} 4 & 2 & 2 \\ 2 & 3 & -5 \\ 3 & 1 & 3 \end{matrix}\right| \begin{matrix} 4 & 2 \\ 2 & 3 \\ 3 & 1 \end{matrix} \Bigg| = \begin{matrix} 36 - 30 + 4 \\ -18 + 20 - 12 \end{matrix} = 0$

$\Rightarrow \{X_1, X_2, X_3\}$ LD.

THM: Let $v_1, \cdots, v_n$ be vectors in a vector space V. A vector $v \in \text{Span}\{v_1, \cdots, v_n\}$ can be uniquely written as a linear combination of $v_1, \cdots, v_n$ if and only if $\{v_1, \cdots, v_n\}$ is LI.

Proof: If $v = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$.

then $(a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n = 0$

$\{v_1, \cdots, v_n\}$ LI $\Leftrightarrow a_1 - b_1 = 0, \cdots, a_n - b_n = 0 \Rightarrow a_1 = b_1, \cdots, a_n = b_n$.

$\Rightarrow$ coefficients in the linear combination are unique, and called the coordinates of v with respect to $\{v_1, \cdots, v_n\}$.

Ex: Check if the vectors are LI.

$p_1(x) = x^2 - 2x + 3$, $p_2(x) = 2x^2 + x + 8$, $p_3(x) = x^2 + 8x + 7$.

check if the linear system has a nonzero solution.

$$\begin{aligned} 0 &= c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x) \\ &= c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) \\ &= x^2(c_1 + 2c_2 + c_3) + x(-2c_1 + c_2 + 8c_3) + 1(3c_1 + 8c_2 + 7c_3) \end{aligned}$$

$$\Leftrightarrow \begin{aligned} x^2&: c_1 + 2c_2 + c_3 = 0 \\ x&: -2c_1 + c_2 + 8c_3 = 0 \\ 1&: 3c_1 + 8c_2 + 7c_3 = 0 \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ has nonzero solution } \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}?$$

check $\begin{vmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 3 & 8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 5 & 10 \\ 0 & 2 & 4 \end{vmatrix} = 0$ $\Leftrightarrow$ nonzero solution $(c_1, c_2, c_3)^T$

$\Leftrightarrow p_1(x), p_2(x), p_3(x)$ are LD.

* The vector space $C^{(n-1)}[a,b]$.

Given $f_1, \cdots, f_n \in C^{(n-1)}[a,b]$. Set

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad x \in [a,b].$$

$$\frac{d}{dx}: \quad c_1 f_1'(x) + c_2 f_2'(x) + \cdots + c_n f_n'(x) = 0$$

$$\vdots \qquad \cdots$$

$$\frac{d^{(n-1)}}{dx^{(n-1)}}: \quad c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0.$$

It leads to a linear system of the form

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \cdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Denote the Wronskian of $f_1, \cdots, f_n$ by

$$W[f_1, \cdots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \cdots & & & \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

then we have

<u>THM</u>: Let $f_1, \cdots, f_n$ be n functions in $C^{(n-1)}[a,b]$.
If there is $x_0 \in [a,b]$ such that $W[f_1, \cdots, f_n](x_0) \neq 0$, then $f_1, f_2, \cdots, f_n$ are LI.

<u>Ex</u>: $e^x, e^{-x} \in C(-\infty, +\infty)$ are LI.

$$W[e^x, e^{-x}] = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2 \neq 0. \Rightarrow LI.$$

<u>Ex</u>: $\sin x, \cos x$ are LI.

$$W[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0 \Rightarrow LI.$$

<u>Ex</u>. Consider $1, x, x^2, x^3$ in $P(-\infty, +\infty)$.

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \neq 0 \Rightarrow LI.$$