9. In each of the following, determine the subspace of $\mathbb{R}^{2\times 2}$ consisting of all matrices that commute with the given matrix:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

10. Let A be a particular vector in $\mathbb{R}^{2\times 2}$. Determine whether the following are subspaces of $\mathbb{R}^{2\times 2}$:

(a) $S_1 = \{B \in \mathbb{R}^{2\times 2} \mid BA = O\}$

(b) $S_2 = \{B \in \mathbb{R}^{2\times 2} \mid AB \neq BA\}$

(c) $S_3 = \{B \in \mathbb{R}^{2\times 2} \mid AB + B = O\}$

11. Determine whether the following are spanning sets for $\mathbb{R}^2$:

(a) $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$ **(b)** $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

12. Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers.

(a) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T\}$

(b) $\{(1, 0, 0)^T, (0, 1, 1)^T, (1, 0, 1)^T, (1, 2, 3)^T\}$

(c) $\{(2, 1, -2)^T, (3, 2, -2)^T, (2, 2, 0)^T\}$

(d) $\{(2, 1, -2)^T, (-2, -1, 2)^T, (4, 2, -4)^T\}$

(e) $\{(1, 1, 3)^T, (0, 2, 1)^T\}$

13. Given

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} 2 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -9 \\ -2 \\ 5 \end{bmatrix}$$

(a) Is $\mathbf{x} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

(b) Is $\mathbf{y} \in \text{Span}(\mathbf{x}_1, \mathbf{x}_2)$?

Prove your answers.

14. Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ be a spanning set for a vector space V.

(a) If we add another vector, $\mathbf{x}_{k+1}$, to the set, will we still have a spanning set? Explain.

(b) If we delete one of the vectors, say $\mathbf{x}_k$, from the set, will we still have a spanning set? Explain.

15. In $\mathbb{R}^{2\times 2}$, let

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Show that $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbb{R}^{2\times 2}$.

16. Which of the sets that follow are spanning sets for P_3? Justify your answers.

(a) $\{1, x^2, x^2 - 2\}$ **(b)** $\{2, x^2, x, 2x + 3\}$

(c) $\{x + 2, x + 1, x^2 - 1\}$ **(d)** $\{x + 2, x^2 - 1\}$

17. Let S be the vector space of infinite sequences defined in Exercise 15 of Section 1. Let S_0 be the set of $\{a_n\}$ with the property that $a_n \to 0$ as $n \to \infty$. Show that S_0 is a subspace of S.

18. Prove that if S is a subspace of $\mathbb{R}^1$, then either $S = \{\mathbf{0}\}$ or $S = \mathbb{R}^1$.

19. Let A be an $n \times n$ matrix. Prove that the following statements are equivalent:

(a) $N(A) = \{\mathbf{0}\}$. **(b)** A is nonsingular.

(c) For each $\mathbf{b} \in \mathbb{R}^n$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

20. Let U and V be subspaces of a vector space W. Prove that their intersection $U \cap V$ is also a subspace of W.

21. Let S be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_1$ and let T be the subspace of $\mathbb{R}^2$ spanned by $\mathbf{e}_2$. Is $S \cup T$ a subspace of $\mathbb{R}^2$? Explain.

22. Let U and V be subspaces of a vector space W. Define

$$U + V = \{\mathbf{z} \mid \mathbf{z} = \mathbf{u} + \mathbf{v} \text{ where } \mathbf{u} \in U \text{ and } \mathbf{v} \in V\}$$

Show that $U + V$ is a subspace of W.

23. Let S, T, and U be subspaces of a vector space V. We can form new subspaces by using the operations of $\cap$ and $+$ defined in Exercises 20 and 22. When we do arithmetic with numbers, we know that the operation of multiplication distributes over the operation of addition in the sense that

$$a(b + c) = ab + ac$$

It is natural to ask whether similar distributive laws hold for the two operations with subspaces.

(a) Does the intersection operation for subspaces distribute over the addition operation? That is, does

$$S \cap (T + U) = (S \cap T) + (S \cap U)$$