$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12$$

Since $W[1, x, x^2, x^3] \neq 0$, the vectors are linearly independent. ■

SECTION 3 EXERCISES

1. Determine whether the following vectors are linearly independent in $\mathbb{R}^2$:

   (a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

   (c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

   (e) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

   (a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}$

   (e) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

3. For each of the sets of vectors in Exercise 2, describe geometrically the span of the given vectors.

4. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

   (a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

5. Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in a vector space V.

   (a) If we add a vector $\mathbf{x}_{k+1}$ to the collection, will we still have a linearly independent collection of vectors? Explain.

   (b) If we delete a vector, say $\mathbf{x}_k$, from the collection, will we still have a linearly independent collection of vectors? Explain.

6. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

   $$\mathbf{y}_1 = \mathbf{x}_1 + \mathbf{x}_2, \quad \mathbf{y}_2 = \mathbf{x}_2 + \mathbf{x}_3, \quad \mathbf{y}_3 = \mathbf{x}_3 + \mathbf{x}_1$$

   Are $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

7. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^n$ and let

   $$\mathbf{y}_1 = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{y}_2 = \mathbf{x}_3 - \mathbf{x}_2, \quad \mathbf{y}_3 = \mathbf{x}_3 - \mathbf{x}_1$$

   Are $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ linearly independent? Prove your answer.

8. Determine whether the following vectors are linearly independent in P_3:

   (a) $1, x^2, x^2 - 2$ (b) $2, x^2, x, 2x + 3$

   (c) $x + 2, x + 1, x^2 - 1$ (d) $x + 2, x^2 - 1$

9. For each of the following, show that the given vectors are linearly independent in $C[0, 1]$:

   (a) $\cos \pi x, \sin \pi x$ (b) $x^{3/2}, x^{5/2}$

   (c) $1, e^x + e^{-x}, e^x - e^{-x}$ (d) e^x, e^{-x}, e^{2x}

10. Determine whether the vectors $\cos x$, 1, $\sin^2(x/2)$ are linearly independent in $C[-\pi, \pi]$.