and hence they form a basis for V. The matrix S is the transition matrix corresponding to the change from the ordered basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ to $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

In many applied problems, it is important to use the right type of basis for the particular application. You may consider a number of applications involving the *eigenvalues* and *eigenvectors* associated with an $n \times n$ matrix A. The key to solving these types of problems is to switch to a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

SECTION 5 EXERCISES

1. For each of the following, find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$:
   (a) $\mathbf{u}_1 = (1, 1)^T$, $\mathbf{u}_2 = (-1, 1)^T$
   (b) $\mathbf{u}_1 = (1, 2)^T$, $\mathbf{u}_2 = (2, 5)^T$
   (c) $\mathbf{u}_1 = (0, 1)^T$, $\mathbf{u}_2 = (1, 0)^T$

2. For each of the ordered bases $\{\mathbf{u}_1, \mathbf{u}_2\}$ in Exercise 1, find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. Let $\mathbf{v}_1 = (3, 2)^T$ and $\mathbf{v}_2 = (4, 3)^T$. For each ordered basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ given in Exercise 1, find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

4. Let $E = [(5, 3)^T, (3, 2)^T]$ and let $\mathbf{x} = (1, 1)^T$, $\mathbf{y} = (1, -1)^T$, and $\mathbf{z} = (10, 7)^T$. Determine the values of $[\mathbf{x}]_E$, $[\mathbf{y}]_E$, and $[\mathbf{z}]_E$.

5. Let $\mathbf{u}_1 = (1, 1, 1)^T$, $\mathbf{u}_2 = (1, 2, 2)^T$, $\mathbf{u}_3 = (2, 3, 4)^T$.
   (a) Find the transition matrix corresponding to the change of basis from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
   (b) Find the coordinates of each of the following vectors with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$:
   (i) $(3, 2, 5)^T$ (ii) $(1, 1, 2)^T$
   (iii) $(2, 3, 2)^T$

6. Let $\mathbf{v}_1 = (4, 6, 7)^T$, $\mathbf{v}_2 = (0, 1, 1)^T$, $\mathbf{v}_3 = (0, 1, 2)^T$, and let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be the vectors given in Exercise 5.
   (a) Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
   (b) If $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

7. Given

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad S = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$$

   find vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ so that S will be the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

8. Given

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad S = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$$

   find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ so that S will be the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$.

9. Let $[x, 1]$ and $[2x - 1, 2x + 1]$ be ordered bases for P_2.
   (a) Find the transition matrix representing the change in coordinates from $[2x - 1, 2x + 1]$ to $[x, 1]$.
   (b) Find the transition matrix representing the change in coordinates from $[x, 1]$ to $[2x - 1, 2x + 1]$.

10. Find the transition matrix representing the change of coordinates on P_3 from the ordered basis $[1, x, x^2]$ to the ordered basis

$$[1, 1 + x, 1 + x + x^2]$$

11. Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $F = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two ordered bases for $\mathbb{R}^n$, and set

$$U = (\mathbf{u}_1, \ldots, \mathbf{u}_n), \quad V = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

   Show that the transition matrix from E to F can be determined by calculating the reduced row echelon form of $(V|U)$.