any polynomial in $P_2$ will have an antiderivative ... the image of polynomials in $P_3$ under the operator $D$. It then follows that ...

## SECTION 1 EXERCISES

1. Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each linear transformation accomplishes.
   (a) $L(\mathbf{x}) = (-x_1, x_2)^T$ (b) $L(\mathbf{x}) = -\mathbf{x}$
   (c) $L(\mathbf{x}) = (x_2, x_1)^T$ (d) $L(\mathbf{x}) = \frac{1}{2}\mathbf{x}$
   (e) $L(\mathbf{x}) = x_2\mathbf{e}_2$

2. Let $L$ be the linear operator on $\mathbb{R}^2$ defined by
$$L(\mathbf{x}) = (x_1 \cos\alpha - x_2 \sin\alpha,\ x_1 \sin\alpha + x_2 \cos\alpha)^T$$
   Express $x_1$, $x_2$, and $L(\mathbf{x})$ in terms of polar coordinates. Describe geometrically the effect of the linear transformation.

3. Let $\mathbf{a}$ be a fixed nonzero vector in $\mathbb{R}^2$. A mapping of the form
$$L(\mathbf{x}) = \mathbf{x} + \mathbf{a}$$
   is called a *translation*. Show that a translation is not a linear operator. Illustrate geometrically the effect of a translation.

4. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If
$$L((1, 2)^T) = (-2, 3)^T$$
   and
$$L((1, -1)^T) = (5, 2)^T$$
   find the value of $L((7, 5)^T)$.

5. Determine whether the following are linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^2$:
   (a) $L(\mathbf{x}) = (x_2, x_3)^T$ (b) $L(\mathbf{x}) = (0, 0)^T$
   (c) $L(\mathbf{x}) = (1 + x_1, x_2)^T$
   (d) $L(\mathbf{x}) = (x_3, x_1 + x_2)^T$

6. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$:
   (a) $L(\mathbf{x}) = (x_1, x_2, 1)^T$
   (b) $L(\mathbf{x}) = (x_1, x_2, x_1 + 2x_2)^T$
   (c) $L(\mathbf{x}) = (x_1, 0, 0)^T$
   (d) $L(\mathbf{x}) = (x_1, x_2, x_1^2 + x_2^2)^T$

7. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$:
   (a) $L(A) = 2A$ (b) $L(A) = A^T$
   (c) $L(A) = A + I$ (d) $L(A) = A - A^T$

8. Let $C$ be a fixed $n \times n$ matrix. Determine whether the following are linear operators on $\mathbb{R}^{n\times n}$:
   (a) $L(A) = CA + AC$ (b) $L(A) = C^2A$
   (c) $L(A) = A^2C$

9. Determine whether the following are linear transformations from $P_2$ to $P_3$:
   (a) $L(p(x)) = xp(x)$
   (b) $L(p(x)) = x^2 + p(x)$
   (c) $L(p(x)) = p(x) + xp(x) + x^2p'(x)$

10. For each $f \in C[0, 1]$, define $L(f) = F$, where
$$F(x) = \int_0^x f(t)\,dt \qquad 0 \le x \le 1$$
   Show that $L$ is a linear operator on $C[0, 1]$ and then find $L(e^x)$ and $L(x^2)$.

11. Determine whether the following are linear transformations from $C[0, 1]$ into $\mathbb{R}^1$:
   (a) $L(f) = f(0)$ (b) $L(f) = |f(0)|$
   (c) $L(f) = [f(0) + f(1)]/2$
   (d) $L(f) = \left\{\int_0^1 [f(x)]^2\,dx\right\}^{1/2}$

12. Use mathematical induction to prove that if $L$ is a linear transformation from $V$ to $W$, then
$$L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \cdots + \alpha_n L(\mathbf{v}_n)$$