SECTION 3 EXERCISES

1. For each of the following linear operators L on $\mathbb{R}^2$, determine the matrix A representing L with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$ (see Exercise 1 of Section 2) and the matrix B representing L with respect to $\{\mathbf{u}_1 = (1, 1)^T, \mathbf{u}_2 = (-1, 1)^T\}$.:

(a) $L(\mathbf{x}) = (-x_1, x_2)^T$ **(b)** $L(\mathbf{x}) = -\mathbf{x}$

(c) $L(\mathbf{x}) = (x_2, x_1)^T$ **(d)** $L(\mathbf{x}) = \frac{1}{2}\mathbf{x}$

(e) $L(\mathbf{x}) = x_2\mathbf{e}_2$

2. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let L be the linear transformation defined by

$$L(\mathbf{x}) = (-x_1, x_2)^T$$

and let B be the matrix representing L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ [from Exercise 1(a)].

(a) Find the transition matrix S corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Find the matrix A representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$ by computing SBS^{-1}.

(c) Verify that

$$L(\mathbf{v}_1) = a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2$$
$$L(\mathbf{v}_2) = a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2$$

3. Let L be the linear transformation on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}$$

and let A be the standard matrix representation of L (see Exercise 4 of Section 2). If $\mathbf{u}_1 = (1, 1, 0)^T$, $\mathbf{u}_2 = (1, 0, 1)^T$, and $\mathbf{u}_3 = (0, 1, 1)^T$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an ordered basis for $\mathbb{R}^3$ and $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is the transition matrix corresponding to a change of basis from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Determine the matrix B representing L with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ by calculating $U^{-1}AU$.

4. Let L be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$$