and let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find the transition matrix V corresponding to a change of basis from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and use it to determine the matrix B representing L with respect to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

5. Let L be the operator on P_3 defined by

$$L(p(x)) = xp'(x) + p''(x)$$

(a) Find the matrix A representing L with respect to $[1, x, x^2]$.

(b) Find the matrix B representing L with respect to $[1, x, 1 + x^2]$.

(c) Find the matrix S such that $B = S^{-1}AS$.

(d) If $p(x) = a_0 + a_1x + a_2(1 + x^2)$, calculate $L^n(p(x))$.

6. Let V be the subspace of $C[a, b]$ spanned by $1, e^x, e^{-x}$, and let D be the differentiation operator on V.

(a) Find the transition matrix S representing the change of coordinates from the ordered basis $[1, e^x, e^{-x}]$ to the ordered basis $[1, \cosh x, \sinh x]$. $[\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.]

(b) Find the matrix A representing D with respect to the ordered basis $[1, \cosh x, \sinh x]$.

(c) Find the matrix B representing D with respect to $[1, e^x, e^{-x}]$.

(d) Verify that $B = S^{-1}AS$.

7. Prove that if A is similar to B and B is similar to C, then A is similar to C.

8. Suppose that $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Show that $A\mathbf{s}_i = \lambda_i \mathbf{s}_i$, $i = 1, \ldots, n$.

(b) Show that if $\mathbf{x} = \alpha_1\mathbf{s}_1 + \alpha_2\mathbf{s}_2 + \cdots + \alpha_n\mathbf{s}_n$, then

$$A^k\mathbf{x} = \alpha_1\lambda_1^k\mathbf{s}_1 + \alpha_2\lambda_2^k\mathbf{s}_2 + \cdots + \alpha_n\lambda_n^k\mathbf{s}_n$$

(c) Suppose that $|\lambda_i| < 1$ for $i = 1, \ldots, n$. What happens to $A^k\mathbf{x}$ as $k \to \infty$? Explain.

9. Suppose that $A = ST$, where S is nonsingular. Let $B = TS$. Show that B is similar to A.

10. Let A and B be $n \times n$ matrices. Show that if A is similar to B, then there exist $n \times n$ matrices S and T, with S nonsingular, such that

$$A = ST \quad \text{and} \quad B = TS$$

11. Show that if A and B are similar matrices, then $\det(A) = \det(B)$.

12. Let A and B be similar matrices. Show that

(a) A^T and B^T are similar.

(b) A^k and B^k are similar for each positive integer k.

13. Show that if A is similar to B and A is nonsingular, then B must also be nonsingular and A^{-1} and B^{-1} are similar.

14. Let A and B be similar matrices and let λ be any scalar. Show that

(a) $A - \lambda I$ and $B - \lambda I$ are similar.

(b) $\det(A - \lambda I) = \det(B - \lambda I)$.

15. The *trace* of an $n \times n$ matrix A, denoted $\mathrm{tr}(A)$, is the sum of its diagonal entries; that is,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Show that

(a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

(b) if A is similar to B, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.