8. In $C[0, 1]$, with inner product defined by (3), consider the vectors 1 and x.
   (a) Find the angle θ between 1 and x.
   (b) Determine the vector projection $\mathbf{p}$ of 1 onto x and verify that $1 - \mathbf{p}$ is orthogonal to $\mathbf{p}$.
   (c) Compute $\|1 - \mathbf{p}\|$, $\|\mathbf{p}\|$, $\|1\|$ and verify that the Pythagorean law holds.

9. In $C[-\pi, \pi]$ with inner product defined by (6), show that $\cos mx$ and $\sin nx$ are orthogonal and that both are unit vectors. Determine the distance between the two vectors.

10. Show that the functions x and x^2 are orthogonal in P_5 with inner product defined by (5), where $x_i = (i-3)/2$ for $i = 1, \ldots, 5$.

11. In P_5 with inner product as in Exercise 10 and norm defined by

$$\|p\| = \sqrt{\langle p, p\rangle} = \left\{\sum_{i=1}^{5} [p(x_i)]^2\right\}^{1/2}$$

compute
   (a) $\|x\|$ (b) $\|x^2\|$
   (c) the distance between x and x^2

12. If V is an inner product space, show that

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$$

satisfies the first two conditions in the definition of a norm.

13. Show that

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

defines a norm on $\mathbb{R}^n$.

14. Show that

$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$$

defines a norm on $\mathbb{R}^n$.

15. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for each of the following vectors in $\mathbb{R}^3$:
   (a) $\mathbf{x} = (-3, 4, 0)^T$ (b) $\mathbf{x} = (-1, -1, 2)^T$
   (c) $\mathbf{x} = (1, 1, 1)^T$

16. Let $\mathbf{x} = (5, 2, 4)^T$ and $\mathbf{y} = (3, 3, 2)^T$. Compute $\|\mathbf{x} - \mathbf{y}\|_1$, $\|\mathbf{x} - \mathbf{y}\|_2$, and $\|\mathbf{x} - \mathbf{y}\|_\infty$. Under which norm are the two vectors closest together? Under which norm are they farthest apart?

17. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in an inner product space. Show that if $\mathbf{x} \perp \mathbf{y}$, then the distance between $\mathbf{x}$ and $\mathbf{y}$ is

$$\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)^{1/2}$$

18. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space that satisfy the Pythagorean law

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal.

19. In $\mathbb{R}^n$ with inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{y}$$

derive a formula for the distance between two vectors $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$.

20. Let A be a nonsingular $n \times n$ matrix and, for each vector $\mathbf{x}$ in $\mathbb{R}^n$, define

$$\|\mathbf{x}\|_A = \|A\mathbf{x}\|_2 \qquad (11)$$

Show that (11) defines a norm on $\mathbb{R}^n$.

21. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$.

22. Let $\mathbf{x} \in \mathbb{R}^2$. Show that $\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$. [*Hint*: Write $\mathbf{x}$ in the form $x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ and use the triangle inequality.]

23. Give an example of a nonzero vector $\mathbf{x} \in \mathbb{R}^2$ for which

$$\|\mathbf{x}\|_\infty = \|\mathbf{x}\|_2 = \|\mathbf{x}\|_1$$

24. Show that, in any vector space with a norm,

$$\|-\mathbf{v}\| = \|\mathbf{v}\|$$

25. Show that, for any $\mathbf{u}$ and $\mathbf{v}$ in a normed vector space,

$$\|\mathbf{u} + \mathbf{v}\| \ge \big|\, \|\mathbf{u}\| - \|\mathbf{v}\| \,\big|$$

26. Prove that, for any $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V,

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

Give a geometric interpretation of this result for the vector space $\mathbb{R}^2$.

27. The result of Exercise 26 is not valid for norms other than the norm derived from the inner product. Give an example of this in $\mathbb{R}^2$ using $\|\cdot\|_1$.

28. Determine whether the following define norms on $C[a, b]$:
   (a) $\|f\| = |f(a)| + |f(b)|$
   (b) $\|f\| = \int_a^b |f(x)|\, dx$
   (c) $\|f\| = \max_{a\le x\le b} |f(x)|$

29. Let $\mathbf{x} \in \mathbb{R}^n$ and show that
   (a) $\|\mathbf{x}\|_1 \le n\|\mathbf{x}\|_\infty$ (b) $\|\mathbf{x}\|_2 \le \sqrt{n}\, \|\mathbf{x}\|_\infty$

Give examples of vectors in $\mathbb{R}^n$ for which equality holds in parts (a) and (b).