(b) Use part (a) and Theorem 5.2 to find the values of the following integrals:
(i) $\int_{-\pi}^{\pi} \sin^4 x \cos x \, dx$
(ii) $\int_{-\pi}^{\pi} \sin^4 x \cos 2x \, dx$
(iii) $\int_{-\pi}^{\pi} \sin^4 x \cos 3x \, dx$
(iv) $\int_{-\pi}^{\pi} \sin^4 x \cos 4x \, dx$

10. Write out the Fourier matrix F_8. Show that $F_8 P_8$ can be partitioned into block form:

$$\begin{pmatrix} F_4 & D_4 F_4 \\ F_4 & -D_4 F_4 \end{pmatrix}$$

11. Prove that the transpose of an orthogonal matrix is an orthogonal matrix.

12. If Q is an $n \times n$ orthogonal matrix and $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then how does the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ compare with the angle between $\mathbf{x}$ and $\mathbf{y}$? Prove your answer.

13. Let Q be an $n \times n$ orthogonal matrix. Use mathematical induction to prove each of the following:
(a) $(Q^m)^{-1} = (Q^T)^m = (Q^m)^T$ for any positive integer m.
(b) $\|Q^m \mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x} \in \mathbb{R}^n$.

14. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that H is both orthogonal and symmetric and hence is its own inverse.

15. Let Q be an orthogonal matrix and let $d = \det(Q)$. Show that $|d| = 1$.

16. Show that the product of two orthogonal matrices is also an orthogonal matrix. Is the product of two permutation matrices a permutation matrix? Explain.

17. How many $n \times n$ permutation matrices are there?

18. Show that if P is a symmetric permutation matrix, then $P^{2k} = I$ and $P^{2k+1} = P$.

19. Show that if U is an $n \times n$ orthogonal matrix, then

$$\mathbf{u}_1\mathbf{u}_1^T + \mathbf{u}_2\mathbf{u}_2^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T = I$$

20. Use mathematical induction to show that if an $n \times n$ matrix Q is both upper triangular and orthogonal, then $\mathbf{q}_j = \pm\mathbf{e}_j$, $j = 1, \ldots, n$.

21. Let

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(a) Show that the column vectors of A form an orthonormal set in $\mathbb{R}^4$.
(b) Solve the least squares problem $A\mathbf{x} = \mathbf{b}$ for each of the following choices of $\mathbf{b}$:
(i) $\mathbf{b} = (4, 0, 0, 0)^T$
(ii) $\mathbf{b} = (1, 2, 3, 4)^T$
(iii) $\mathbf{b} = (1, 1, 2, 2)^T$

22. Let A be the matrix given in Exercise 21.
(a) Find the projection matrix P that projects vectors in $\mathbb{R}^4$ onto $R(A)$.
(b) For each of your solutions $\mathbf{x}$ to Exercise 21(b), compute $A\mathbf{x}$ and compare it with $P\mathbf{b}$.

23. Let A be the matrix given in Exercise 21.
(a) Find an orthonormal basis for $N(A^T)$.
(b) Determine the projection matrix Q that projects vectors in $\mathbb{R}^4$ onto $N(A^T)$.

24. Let A be an $m \times n$ matrix, let P be the projection matrix that projects vectors in $\mathbb{R}^m$ onto $R(A)$, and let Q be the projection matrix that projects vectors in $\mathbb{R}^n$ onto $R(A^T)$. Show that
(a) $I - P$ is the projection matrix from $\mathbb{R}^m$ onto $N(A^T)$.
(b) $I - Q$ is the projection matrix from $\mathbb{R}^n$ onto $N(A)$.

25. Let P be the projection matrix corresponding to a subspace S of $\mathbb{R}^m$. Show that
(a) $P^2 = P$ **(b)** $P^T = P$

26. Let A be an $m \times n$ matrix whose column vectors are mutually orthogonal, and let $\mathbf{b} \in \mathbb{R}^m$. Show that if $\mathbf{y}$ is the least squares solution of the system $A\mathbf{x} = \mathbf{b}$, then

$$y_i = \frac{\mathbf{b}^T\mathbf{a}_i}{\mathbf{a}_i^T\mathbf{a}_i} \qquad i = 1, \ldots, n$$

27. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\| \le \|\mathbf{v}\|$. Under what conditions does equality occur?

28. Let $\mathbf{v}$ be a vector in an inner product space V and let $\mathbf{p}$ be the projection of $\mathbf{v}$ onto an n-dimensional subspace S of V. Show that $\|\mathbf{p}\|^2 = \langle \mathbf{p}, \mathbf{v} \rangle$.

29. Consider the vector space $C[-1, 1]$ with inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

and norm

$$\|f\| = (\langle f, f \rangle)^{1/2}$$

(a) Show that the vectors 1 and x are orthogonal.