Part I: The Ratio Test

1.

What does the Ratio Test say?

If $\lim_{n} \left| \frac{a_{n+1}}{a_n} \right| = \ell$

$\ell < 1 \Rightarrow \sum a_n$ is Absolutely convergent

$\ell > 1 \Rightarrow \sum a_n$ is divergent

$\ell = 1 \Rightarrow$ Test is inconclusive

2.

Determine whether the series $\sum_{k=1}^{\infty} \frac{k}{5^k}$ converges or diverges.

Ratio Test:

$$\lim_{k\to\infty} \frac{k+1}{\cancel{5^{k+1}}} \cdot \frac{\cancel{5^k}}{k} = \lim_{k} \frac{1}{5}\left(\frac{k+1}{k}\right) = \frac{1}{5} < 1$$

The series converges by the Ratio Test.

3.

Determine whether the series $\sum_{k=1}^{\infty}(-1)^k \frac{3^k k^3}{5^k}$ converges or diverges.

Ratio Test.

$$\lim_{k\to\infty}\left|\frac{3^{\cancel{k+1}}(k+1)^3}{5^{\cancel{k+1}}}\cdot\frac{\cancel{5^k}}{\cancel{3^k}k^3}\right|$$

$$\lim_{k}\frac{3}{5}\frac{(k+1)^3}{k^3} = \frac{3}{5}\lim\cancel{\left(\frac{k+1}{k}\right)^3} \to 1 = \frac{3}{5} < 1$$

The series absolutely converges by Ratio Test.

4.

Determine whether the series $\sum_{k=1}^{\infty}(-1)^k \frac{5^k}{3^k k^5}$ converges or diverges.

Ratio Test

$$\lim_{k\to\infty}\left|\frac{5^{\cancel{k+1}}}{3^{\cancel{k+1}}(k+1)^5}\cdot\frac{\cancel{3^k}\,k^5}{\cancel{5^k}}\right| = \frac{5}{3}\lim\left(\frac{k}{k+1}\right)^5 = \frac{5}{3} > 1$$

so the series diverges by R.T.

5.

Determine whether the series $\sum_{k=1}^{\infty} \frac{(2k+1)!\cos k}{k^5 10^k}$ converges or diverges.

Claim: the series

$$\lim_{k\to\infty} \frac{(2k+1)!\cos k}{k^5\, 10^k} = \lim_{k} \frac{1\cdot 2\cdot 3\cdot \ldots \cdot k\cdot \overbrace{(k+1)\cdots 2k}\,(2k+1)}{k\,k\,k\,k\,k\ \underbrace{10}_{\infty}\cdots \underbrace{10}_{\infty}\ 1}(\cos k)$$

$$(\cos k)\,\frac{(1\cdot 2\cdot 3\cdots (k-1))(2k+1)\cdots}{k\cdot k\cdot k} \not\to 0$$

The series diverges by test for divergence.

$3(k+1)+1$

6.

Determine whether the series $\sum_{k=1}^{\infty} \frac{k5^{2k}}{(3k+1)!}$ converges or diverges.

Ratio Test.

$1 \cdot 2 \cdots 3k+1$ / $3k+1 \quad 3k+2 \quad 3k+4$

$$\lim_{k} \left| \frac{(k+1)\, 5^{\cancel{2k+2}}}{(3k+4)!} \cdot \frac{(3k+1)!}{k\, \cancel{5^{2k}}} \right|$$

$$\lim_{k} \frac{5^2}{(3k+2)(3k+3)(3k+4)} \left(\frac{k+1}{k}\right) \to 0$$

conv. by R.T.

7.

If $a_1 = 2$ and $a_{k+1} = \frac{5k-17\sin k}{4k+3} a_k$ determine whether $\sum a_k$ converges or diverges.

Ratio Test (RT)

$$\lim_{k} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k} \left| \frac{(5k - 17\sin k)\cancel{a_k}}{4k+3} \cdot \frac{1}{\cancel{a_k}} \right|$$

$$\lim_{k} \left(\frac{5k}{4k+3} - \frac{17\sin k}{4k+3}\right) = \frac{5}{4} > 1$$

$$\underset{\substack{\downarrow \\ 0}}{\frac{-17}{4k+3}} \le \underset{\substack{\downarrow \\ 0}}{\frac{17\sin k}{4k+3}} \le \underset{\substack{\downarrow \\ 0}}{\frac{17}{4k+3}}$$

$\boxed{\text{diverg. by R.T}}$

8.

If $a_1 = 2$ and $a_{k+1} = (k \sin \frac{1}{2k})a_k$ determine whether $\sum a_k$ converges or diverges.

R.T.

$$\lim_{k\to\infty} \left| \frac{(k \sin \frac{1}{2k})\cancel{a_k}}{\cancel{a_k}} \right| = \infty \cdot 0$$

$$= \lim_{k} \frac{\sin \frac{1}{2k}}{\frac{1}{k}} = \frac{0}{0} \text{ L'H.R.}$$

$$= \lim \frac{\cancel{-\frac{1}{2k^2}} \; \cancel{\cos \frac{1}{2k}}^{\to \cos 0}}{\cancel{-\frac{1}{k^2}}} = \frac{1}{2} \cdot 1 < 1$$

Conv. by R.T.

Part II: Power Series

Problem 1. What is a power series centered at a? What is the *radius of convergence* and *interval of convergence* of a power series?

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

c_n in $\mathbb{R}$

I is either $\{a\}$ or $(-\infty, +\infty)$ or any combination of $[a-R, a+R)$, $(a-R, a+R]$, $(a-R, a+R)$, $[a-R, a+R]$

R = half of the length of I.

Problem 2. Write $f(x) = \frac{x}{1+x}$ as a power series. What is a? What is c_k?

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_0^{\infty} (-x)^n = \boxed{\sum_0^{\infty} (-1)^n x^n}$$

$$\frac{x}{1+x} = \underbrace{x \sum_0^{\infty} (-1)^n x^n} = \sum_{n=0}^{\infty} (-1)^n x^{n+1}$$

$$C_n = (-1)^n$$

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Problem 3. Complete the theorem: For a given power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ there are only three possibilities:

(1) series only converges at a

$R = 0 \quad I = \{a\}$

(2) series converges in $I = (-\infty, +\infty)$

$R = \infty$

(3) series converges in $I = (a-R, a+R)$

any combination of open/closed.

Problem 4. Find the ROC and IOC for $\sum_{k=0}^{\infty} \frac{(-3)^k x^k}{\sqrt{k+1}}$.

center = 0

R.T.

$$\lim_{k} \left| \frac{3^{\cancel{k+1}} x^{\cancel{k+1}}}{\sqrt{k+2}} \cdot \frac{\sqrt{k+1}}{\cancel{3^k}\,\cancel{x^k}} \right| =$$

$$= \lim_{k} 3 \underbrace{\frac{\sqrt{k+1}}{\sqrt{k+2}}}_{\to 1} |x| = 3|x| < 1 \text{ so}$$

$$|x| < \frac{1}{3}$$

$$-\frac{1}{3} < x < \frac{1}{3} \Rightarrow R = \frac{1}{3}$$

$$x = -\frac{1}{3} \Rightarrow \sum \frac{(-3)^k \left(-\frac{1}{3}\right)^k}{\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \approx \sum_{1}^{\infty} \frac{1}{\sqrt{k}}$$

div. by p-series $p = \frac{1}{2} < 1$

$$x = \frac{1}{3} \Rightarrow \sum \frac{(-3)^k \left(\frac{1}{3}\right)^k}{\sqrt{k+1}} = \sum_{0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$$

converges by AST.

$$R = \frac{1}{3} \qquad I = \left(-\frac{1}{3}, \frac{1}{3}\right]$$

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Problem 5. Find the ROC and IOC for $\sum_{k=0}^{\infty} \frac{k(x+2)^k}{3^{k+1}}$. center $=-2$

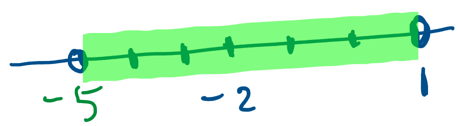
R.T

$$\lim_{k} \left| \frac{(k+1)(x+2)^{\cancel{k+1}}}{3^{\cancel{k+2}}} \cdot \frac{\cancel{3^{k+1}}}{k\cancel{(x+2)^k}} \right| =$$

$$\lim_{k} \frac{1}{3} \underbrace{\frac{k+1}{k}}_{\to 1} |x+2| = \frac{1}{3}|x+2| < 1$$

so $|x+2| < 3$

$$-3 < x+2 < 3$$
$$-2 \qquad -2 \qquad -2$$
$$-5 < x < 1$$



$$\boxed{R = 3}$$

$$x=-5 \Rightarrow \sum \frac{k(-3)^k}{3^{k+1}} = \sum (-1)^k \frac{k}{3} =$$

$$= \frac{1}{3} \boxed{\sum (-1)^k (k)} \text{ divergent.}$$

$$x=1 \Rightarrow \sum \frac{k(3)^k}{3^{k+1}} = \sum \frac{k}{3} = \frac{1}{3}\sum k$$

diverg

3^{k+1} diverg.

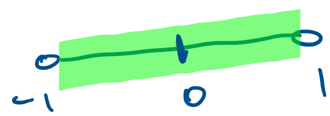
$I = (-5, 1)$

Problem 6. Find the ROC and IOC for $\sum_{k=0}^{\infty}(-1)^k k x^k$.

RT

$$\lim_{k} \left| \frac{(k+1)\,\cancel{x^{k+1}}}{k\,\cancel{x^k}} \right| = \lim |x| \cancel{\left(\frac{k+1}{k}\right)}^{\to 1} = |x|$$

$|x| < 1$



$\boxed{R = 1}$

$x = -1 \Rightarrow \sum (-1)^k k (-1)^k = \sum k$ $\boxed{\text{div.}}$

$x = 1 \Rightarrow \sum (-1)^k k\, 1^k = \sum (-1)^k k$ div.

$\boxed{I = (-1, 1).}$

$\{a_k\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

Problem 7. Find the ROC and IOC for $\sum_{k=0}^{\infty}(-1)^k a_k x^k$, where $a_0 = 1$ and $\underline{a_{k+1} = \frac{k}{k+1}a_k}$.

$\frac{2}{3} \frac{1}{2}$

$\frac{3}{4} \frac{1}{3}$

$$\lim_{k} \left| \frac{\cancel{(-1)^{k+1}}\left(\frac{k}{k+1}\right)\cancel{a_k}\, \cancel{x^{k+1}}}{\cancel{(-1)^k}\,\cancel{a_k}\,\cancel{x^k}} \right| =$$

(-1)

$$\lim |x| \frac{k}{k+1} = |x| < 1$$

$-1 < x < 1$ $\boxed{R = 1}$

$x = -1 \Rightarrow \sum (-1)^k a_k (-1)^k = \sum a_k$ $\boxed{\text{div.}}$

$x = 1 \Rightarrow \sum (-1)^k a_k 1^k = \sum (-1)^k a_k$

$\sum (-1)^k \frac{1}{n}$ conv. by AST.

$$\boxed{I = (-1, 1].}$$

Problem 8. Find the ROC and IOC for $\sum_{k=0}^{\infty}(-1)^k k x^k$.

Already done Pb6.

Problem 9. Find the ROC and IOC for $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k5^k}x^k$.

R.T.

$$\lim_{k} \left| \frac{x^{\cancel{k+1}}}{(k+1)5^{\cancel{k+1}}} \cdot \frac{k\cancel{5^k}}{\cancel{x^k}} \right| =$$

$$= \lim_{k} \frac{|x|}{5} \frac{k}{k+1} \to 1 = \frac{|x|}{5} < 1 \text{ so } |x| < 5$$

$-5 < x < 5$ $\qquad \boxed{R = 5}$

$$x = -5 \Rightarrow \sum_{1}^{\infty} (-1)^k \frac{1}{k\cancel{5^k}} \cancel{(-5)^k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

diverg. HARMONIC.

$$x = 5 \Rightarrow \sum_{1}^{\infty} (-1)^k \frac{1}{k\cancel{5^k}} \cancel{5^k} =$$

$= \sum \frac{(-1)^k}{k}$ conv. by AST.

$$I = [-5, 5]$$

Problem 10. Find the ROC and IOC for $\sum_{k=1}^{\infty} \frac{(2x-1)^k}{5^k\sqrt{k}}$. center $= \frac{1}{2}$

R.T.

$$\lim_{k} \left| \frac{(2x-1)^{\cancel{k+1}}}{5^{\cancel{k+1}}\sqrt{k+1}} \cdot \frac{\cancel{5^k}\sqrt{k}}{\cancel{(2x-1)^k}} \right|$$

$$\lim \cancel{\sqrt{\frac{k}{k+1}}}^{\to 1} \frac{|2x-1|}{5} = \frac{1}{5}|2x-1| < 1$$

$$|2x-1| < 5$$

$$-5 < 2x-1 < 5$$
$$+1 \quad +1 \quad +1$$

$$\frac{-4}{2} < \frac{2x}{2} < \frac{6}{2}$$

$$-2 < x < 3$$

-2 $\frac{1}{2}$ 3

$$R = \frac{3+2}{2} = 2.5$$

$$\boxed{R = \frac{5}{2}}$$

$x = -2 \Rightarrow \sum \frac{(-5)^k}{5^k\sqrt{k}} = \sum_{1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ conv. by AST.

$x = 3 \Rightarrow \sum \frac{5^k}{5^k\sqrt{k}} = \sum \frac{1}{\sqrt{k}}$ div.

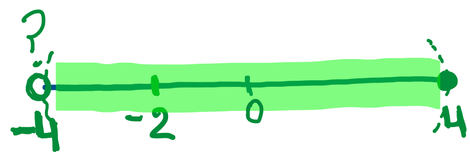
p-series $p = \frac{1}{2} < 1$.

$$I = [-2, 3)$$

Problem 11. Assume that $\sum_{k=0}^{\infty} c_k 4^k$ converges. What can we say about:

(1) $\sum_{k=0}^{\infty} c_k (-2)^k$.

conv.

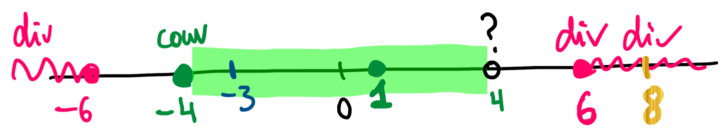


(2) $\sum_{k=0}^{\infty} c_k (-4)^k$. we don't know
not enough information.

Problem 12. Assume that $\sum_{k=0}^{\infty} c_k x^k$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of:

(1) $\sum_{k=0}^{\infty} c_k$ $\Rightarrow x = 1$

convergent.



(2) $\sum_{k=0}^{\infty} c_k 8^k$

divergent.

(3) $\sum_{k=0}^{\infty} c_k (-3)^k$

convergent.

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