Solutions to Week 1 in Review

courtesy: David J. Manuel

(covering 151 Review and 5.5)

(Problems with a * beside them will also be done in Python)

1 MATH 151 Review

1. Find $\frac{d}{dx}\int_0^{x^2} e^{-t^2}\,dt$.

Pt I Fund Theorem of Calculus

If f is a continuous function and $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Simpler Ex $\frac{d}{dx}\int_0^x e^{-t^2}\,dt = \boxed{e^{-x^2}}$

inner function

Chain Rule $\frac{d}{dx}\int_0^{x^2} e^{-t^2}\,dt = \boxed{e^{-(x^2)^2}\cdot 2x} = \boxed{2x\,e^{-x^4}}$

$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))g'(x)$

2. Evaluate $\lim_{x\to\infty} \frac{2}{x\ln(x)} \int_1^x \ln(t)\,dt$

$$= \lim_{x\to\infty} \frac{2\int_1^x \ln(t)\,dt}{x\ln(x)} \quad \frac{\infty}{\infty}$$ L'Hospital's Rule

$$= \lim_{x\to\infty} \frac{2\ln(x)}{x\left(\frac{1}{x}\right)+(1)\ln x} = \lim_{x\to\infty} \frac{2\ln(x)}{1+\ln(x)} \quad \frac{\infty}{\infty}$$ L'Hospital again OR

$$= \lim_{x\to\infty} \frac{\cancel{\ln(x)}(2)}{\cancel{\ln(x)}\left(\frac{1}{\ln(x)}+1\right)} = \boxed{2}$$

0

Recall Pt 2 FTC

If F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

3. For what value(s) of b does $\int_0^b (6x^{1+1} - 1)\, dx = 7b$?*

Evaluate by FTC pt 2

Recall Power Rule:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int m \, dx = mx + C$$

$$\frac{6}{2}x^2 - x \Big|_0^b = 7b$$

Evaluated at...

$$(3b^2 - b) - (3 \cdot 0^2 - 0) = 7b$$

$$3b^2 - b = 7b$$
$$-7b \quad -7b$$
$$3b^2 - 8b = 0$$

Steps by hand:

1) integrate left hand side
2) solve equation

$$b(3b - 8) = 0$$

$$b = 0 \qquad 3b - 8 = 0$$
$$+8 \quad +8$$
$$\frac{3b}{3} = \frac{8}{3}$$
$$b = \frac{8}{3}$$

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from sympy import *
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```
x=symbols('x')
f=6*x-1
b=symbols('b')   # Can combine both:  x,b=symbols('x b')
LHS=integrate(f,(x,0,b))
eqn=LHS-7*b
bsoln=solve(eqn,b)
print('The values of b which solve the equation are',bsoln)
```

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The values of b which solve the equation are [0, 8/3]
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4. Evaluate the following integrals:

(a) $\int_{-3}^{2}(6-y-y^2)\,dy$*

$= 6y - \frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_{-3}^{2}$

$= \boxed{\left(6(2) - \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3\right) - \left(6(-3) - \frac{1}{2}(-3)^2 - \frac{1}{3}(-3)^3\right)}$

OK on workout? Ask your instructor.

$= 12 - 2 - \frac{8}{3} + (+18) + \frac{9}{2} + (-9)$

$= 19 - \frac{8}{3} + \frac{9}{2}$

$\frac{114}{6} - \frac{16}{6} + \frac{27}{6} = \boxed{\frac{125}{6}}$

Steps to Solve
1) integrate
2) substitute numbers
3) subtract

```
from sympy import *
```

```
y=symbols('y')
f=6-y-y**2
# Method I: by applying FTC F(b)-F(a)
F=integrate(f,y)  # Can also say F=f.integrate(y)
defint=F.subs(y,2)-F.subs(y,-3)
print('The definite integral evaluates to',defint)

# Method II: directly integrate in Python
defint2=f.integrate((y,-3,2))
print('Evaluating directly in Python yields',defint2)
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```
The definite integral evaluates to 125/6
Evaluating directly in Python yields 125/6
```

(b) $\int_0^1 (2-x)(8x^3)\,dx$ multiply out

$= \int_0^1 (16x^{3+1} - 8x^{4+1})\,dx$

$= \frac{16}{4}x^4 - \frac{8}{5}x^5 \Big|_0^1$

$= \left(4(1)^4 - \frac{8}{5}(1)^5\right) - \left(4(0)^4 - \frac{8}{5}(0)^5\right)$?

$= 4 - \frac{8}{5} = \boxed{\frac{12}{5}}$

(c) $\int (\tan x)^2\, dx$ (HINT: an identity is helpful here...)

$= \int \left((\sec x)^2 - 1 \right) dx$

$= \boxed{\tan x - x + C}$

$\tan^2 x$

Recall :

$\frac{(\sin x)^2}{(\cos x)^2} + \frac{(\cos x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2}$

$(\tan x)^2 + 1 = (\sec x)^2$

$(\tan x)^2 = (\sec x)^2 - 1$

$\frac{d}{dx}(\tan x) = (\sec x)^2$

So $\int (\sec x)^2\, dx = \tan x + C$

(d) $\int_{-1}^{1} (2 - |x| - x^4)\, dx$

Three strategies:

Method I

defn of $|x| = \begin{cases} -1x & \text{if } x<0 \\ x & \text{if } x \geq 0 \end{cases}$

$$= \int_{-1}^{0} (2 - (-1x) - x^4)\, dx + \int_{0}^{1} (2 - x - x^4)\, dx$$

$$= \int_{-1}^{0} (2 + x - x^4)\, dx \quad + \quad 2x - \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5 \Big|_0^1$$

$$= 2x + \tfrac{1}{2}x^2 - \tfrac{1}{5}x^5 \Big|_{-1}^{0} \quad +$$

$$= \boxed{(0 + 0 - 0) - (-2 + \tfrac{1}{2} + \tfrac{1}{5}) + (2 - \tfrac{1}{2} - \tfrac{1}{5}) - (0 - 0 - 0)}$$

$$= \quad 2 - \tfrac{1}{2} - \tfrac{1}{5} + 2 - \tfrac{1}{2} - \tfrac{1}{5}$$

$$= \quad 3 - \tfrac{2}{5} = \boxed{\tfrac{13}{5}}$$

Method II

Area of triangles

$$= \int_{-1}^{1} 2\, dx - \int_{-1}^{1} |x|\, dx - \int_{-1}^{1} x^4\, dx$$

$$= 2x \Big|_{-1}^{1} - \left(\tfrac{1}{2}(1)(1) + \tfrac{1}{2}(1)(1)\right) - \left(\tfrac{1}{5}x^5 \Big|_{-1}^{1}\right)$$

$$= 2 - (-2) - \left(\tfrac{1}{2} + \tfrac{1}{2}\right) - \left(\tfrac{1}{5} + \tfrac{1}{5}\right)$$

Method III: Use symmetry

If f is an even function, $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

always positive

$$= 2\int_{0}^{1} (2 - |x| - x^4)\, dx$$

$$= 2\int_{0}^{1} (2 - x - x^4)\, dx$$

$\vdots$

Substitution : $\int f'(g(x))g'(x)\,dx = f(g(x)) + C$ by letting $u = g(x)$

"stuff in parentheses which is INSIDE another function"

2 Section 5.5

function in parentheses inside another function AND derivative of stuff in integral

1. Evaluate the following integrals:

(a) $\displaystyle\int_0^{\ln(5)} \frac{e^x}{1+e^x}\,dx = \int_0^{\ln(5)} e^x(1+e^x)^{-1}\,dx$

Let $u = 1+e^x$

$du = e^x\,dx$

if $x=\ln(5)$, $u = 1+e^{\ln(5)} = 6$

if $x = 0$, $u = 1+e^0 = 2$

Method I

leave original boundaries and return to original variable

$$= \int_{x=0}^{x=\ln(5)} u^{-1}\,du$$

$$= \ln|u| \Big|_{x=0}^{x=\ln(5)}$$

$$= \ln|1+e^x| \Big|_0^{\ln 5}$$

$$= \ln|1+\cancel{e}^{\ln 5}| - \ln|1+e^0|$$

1

$$= \boxed{\ln(6) - \ln(2)} = \ln(3)$$

Method II change the boundaries

$$= \int_2^6 u^{-1}\,du$$

$$= \ln u \Big|_2^6$$

$$= \ln(6) - \ln(2)$$

stuff in parentheses inside another function
AND derivative of stuff in integral

Differentials (3.11)
" $dy = f'(x)\,dx$ "

(b) $\int (\sin(\theta))^3 \cos(\theta)\, d\theta$

Let $u = \sin(\theta)$
Then $du = \cos(\theta)\, d\theta$

$\int u^3 \, du$

$= \frac{1}{4} u^4 + C$

$= \boxed{\frac{1}{4} (\sin\theta)^4 + C}$

Not exactly deriv of stuff (need to remove the -1)

(c) $\displaystyle\int_0^{\sqrt{3}} \frac{x+1}{x^2+1}\,dx = \int_0^{\sqrt{3}} (x+1)(x^2+1)^{-1}\,dx$

derivative of arctan(x)

$$= \int_0^{\sqrt{3}} \frac{x}{x^2+1}\,dx + \int_0^{\sqrt{3}} \frac{1}{x^2+1}\,dx$$

Let $u = x^2+1$
$du = 2x\,dx$
if $x=\sqrt{3}$, $u = \sqrt{3}^2+1 = 4$
$x=0$, $u = 0^2+1 = 1$

\# ONLY works with numbers

$$= \frac{1}{2}\int_0^{\sqrt{3}} 2x\,(x^2+1)^{-1}\,dx \qquad + \arctan(x)\Big|_0^{\sqrt{3}}$$

$\tan x = \dfrac{\sqrt{3}/2}{1/2}$ $\dfrac{\sin x}{\cos x}$

$$= \frac{1}{2}\int_1^4 u^{-1}\,du \qquad + \arctan(\sqrt{3}) - \arctan(0)$$

$$= \frac{1}{2}\ln|u|\Big|_1^4 \qquad + \frac{\pi}{3} - 0$$

$$= \frac{1}{2}\ln(4) - \frac{1}{2}\ln(1) \qquad + \frac{\pi}{3}$$

($\ln(1) = 0$)

$$= \boxed{\frac{1}{2}\ln(4) + \frac{\pi}{3}}$$

(d) $\displaystyle\int \frac{1}{x(1+(\ln(x))^2)}\,dx$

$1+(\ln(x))^2$ is in parentheses, BUT derivative is NOT inside integral

CANNOT do $\cancel{\dfrac{1}{2\ln(x)}\displaystyle\int \frac{2\ln(x)}{x\,(1+(\ln(x))^2)}}$

Let $u = \ln(x)$

then $du = \frac{1}{x}\,dx$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan(u) + C$$

$$= \boxed{\arctan(\ln(x)) + C}$$

(e) $-\frac{1}{2}\int_0^2 -2x e^{-x^2}\,dx$

Let $u = -x^2$

$du = -2x\,dx$

if $x=2$ $u=-2^2=-4$

if $x=0$ $u=-0^2=0$

$= -\frac{1}{2}\int_0^{-4} e^u\,du$

$= +\frac{1}{2}\int_{-4}^{0} e^u\,du$

$= \frac{1}{2} e^u \Big|_{-4}^{0}$

$= \frac{1}{2}(e^0 - e^{-4}) = \boxed{\frac{1}{2}(1-e^{-4})}$

(f) $\int_0^4 x^3\sqrt{9+x^2}\,dx$

Let $u = 9+x^2$
Then $du = 2x\,dx$

if $x=4$, $u=9+4^2=25$
if $x=0$, $u=9+0^2=9$

$x^2 = u-9$

$$= \frac{1}{2}\int_0^4 2x \cdot x^2\sqrt{9+x^2}\,dx$$

$$= \frac{1}{2}\int_9^{25} x^2\sqrt{u}\,du$$

$$= \frac{1}{2}\int_9^{25}(u-9)u^{1/2}\,du$$

$$= \frac{1}{2}\int_9^{25}\left(u^{3/2+1} - 9u^{1/2+1}\right)du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 9\left(\frac{2}{3}\right)u^{3/2}\right)\Big|_9^{25}$$

$$= \boxed{\frac{1}{2}\left(\frac{2}{5}(25)^{5/2} - 6(25)^{3/2}\right) - \frac{1}{2}\left(\frac{2}{5}(9)^{5/2} - 6(9)^{3/2}\right)}$$