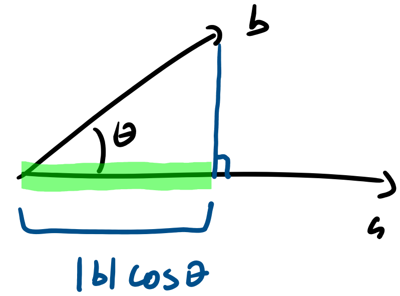
Appendix J.2: The Dot Product

Definition: The **dot product** of two nonzero vectors **a** and **b** is the number

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between the vectors **a** and **b**, $0 \le \theta \le \pi$. If either **a** or **b** is **0**, then we define $\mathbf{a}\cdot\mathbf{b} = 0$.



Example: Find $\mathbf{a}\cdot\mathbf{b}$ if $|\mathbf{a}| = 4$, $|\mathbf{b}| = 10$, and $\theta = \frac{\pi}{6}$

$$\begin{aligned} a\cdot b &= |a|\,|b| \cos\theta \\ &= 4\cdot 10 \cos\tfrac{\pi}{6} \\ &= 40\cdot\tfrac{\sqrt{3}}{2} = 20\sqrt{3} \end{aligned}$$

Example: Explaine what the dot product tells us if $\mathbf{a}\cdot\mathbf{b}$ is positive? is negative?

dot product positive.

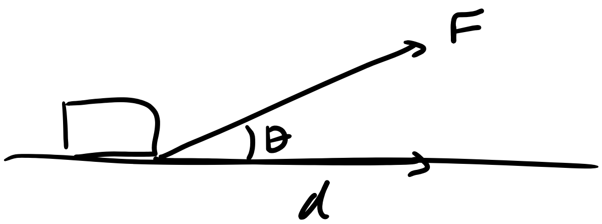
$\cos\theta > 0 \longrightarrow 0 \le \theta < \frac{\pi}{2}$

Both vectors are in the "same ish" direction.

dot product is neg.

$\hookrightarrow \frac{\pi}{2} < \theta \le \pi$

Definition: The **work** done by a force, $\mathbf{F}$, in moving an object from point P to point Q, or with displacement $\mathbf{D} = \overrightarrow{PQ}$, is given by $\mathbf{W} = \mathbf{F} \cdot \mathbf{D}$.



$w = |d| \underbrace{|F| \cos\theta}$

Example: Find the work of using a force of 10N to move a block 3 meters if the force is applied at an angle of 25^o to the ground. (Assume that the ground is level.)

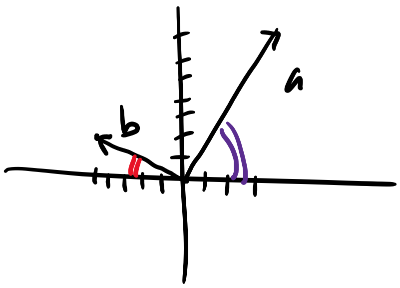
$|F| = 10N \qquad |D| = 3m \qquad \theta = 25^\circ$

$W = |F|\,|D| \cos\theta$

$= 10 \cdot 3 \cdot \cos 25 = 27.19\ N\cdot m = 27.19\ J$

Definition: Two <u>non-zero vectors</u> a and b are perpendicular (orthogonal) if and only if $\underbrace{\mathbf{a} \cdot \mathbf{b} = 0}_{\theta = \frac{\pi}{2}}$

Example: Are the vectors $\mathbf{a} = \langle 3, 7 \rangle$ and $\mathbf{b} = \langle -5, 2 \rangle$ orthogonal?



$$\begin{aligned} a \cdot b &= 3(-5) + 7(2) \\ &= -15 + 14 \\ &= -1 \end{aligned}$$

not perp.

Definition: Alternate definition for the dot product of two vectors. For two vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ then $\mathbf{a} \cdot \mathbf{b} = a_1 * b_1 + a_2 * b_2$

Proof of alternate definition:(optional). Let $(a) = \langle a_1, a_2 \rangle$ and $(b) = \langle b_1, b_2 \rangle$

$$|\mathrm{a} - \mathrm{b}|^2 = |\mathrm{a}|^2 + |\mathrm{b}|^2 - 2|\mathrm{a}||\mathrm{b}| \cos\theta$$

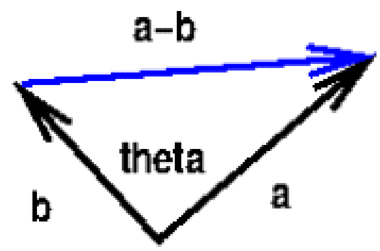
$$|\mathrm{a} - \mathrm{b}|^2 = |\mathrm{a}|^2 + |\mathrm{b}|^2 - 2\mathrm{a} \cdot \mathrm{b}$$

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2\mathrm{a} \cdot \mathrm{b}$$

$$2\mathrm{a} \cdot \mathrm{b} = a_1^2 + a_2^2 + b_1^2 + b_2^2 - (a_1^2 - 2a_1b_1 + b_1^2) - (a_2^2 - 2a_2b_2 + b_2^2)$$

$$2\mathrm{a} \cdot \mathrm{b} = 2a_1b_1 + 2a_2b_2$$

$$\mathrm{a} \cdot \mathrm{b} = a_1b_1 + a_2b_2$$



$a \cdot a = a_1 \cdot a_1 + a_2 \cdot a_2 = (a_1)^2 + (a_2)^2 = \left(\sqrt{a_1^2 + a_2^2}\right)^2$
$= |a|^2$

Properties of the Dot Product: If **a**, **b**, and **c** are vectors and m is a scalar, then

$\mathbf{a} \cdot \mathbf{a} = \lvert\mathbf{a}\rvert^2$	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{0} \cdot \mathbf{a} = 0$
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$	$(m\mathbf{a}) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$	

Definition: The **orthogonal compliment** of $\mathbf{a} = \langle a_1, a_2 \rangle$ is $\mathbf{a}^{\perp} = \langle -a_2, a_1 \rangle$

$a = \langle 3, 10 \rangle \qquad a^{\perp} = \langle -10, 3 \rangle$

$\langle 10, -3 \rangle$ is also perp. But not the orthogonal compliment.

Example: What value(s) of x will make $\langle x, 4 \rangle$ and $\langle x, 7x \rangle$ orthogonal?

$\langle x, 4 \rangle \cdot \langle x, 7x \rangle = 0$

$x^2 + 28x = 0$

$x(x + 28) = 0$

$\underbrace{x = 0}_{\text{silly case.}} \qquad x = -28$

Example: A constant force $\mathbf{F} = 2\mathbf{i} + 4\mathbf{j}$, in Newtons, is used to move an object from $A(2,5)$ to $B(7,9)$. Find the work done if the distance between the points is measured in meters.

$D = \overrightarrow{AB} = \langle 5, 4 \rangle$

$W = F \cdot D = 2(5) + 4(4) = 10 + 16 = 26\ Nm = 26\ J$

Example: Find the angle between $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}$.

$$|a| = \sqrt{3^2 + 5^2} = \sqrt{9+25} = \sqrt{34}$$

$$|b| = \sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20}$$

$$a \cdot b = |a|\,|b| \cos\theta$$

$$3(4) + 5(2) = \sqrt{34}\sqrt{20}\cos\theta$$

$$22 = \sqrt{34}\sqrt{20}\cos\theta$$

$$\cos\theta = \frac{22}{\sqrt{34}\sqrt{20}}$$

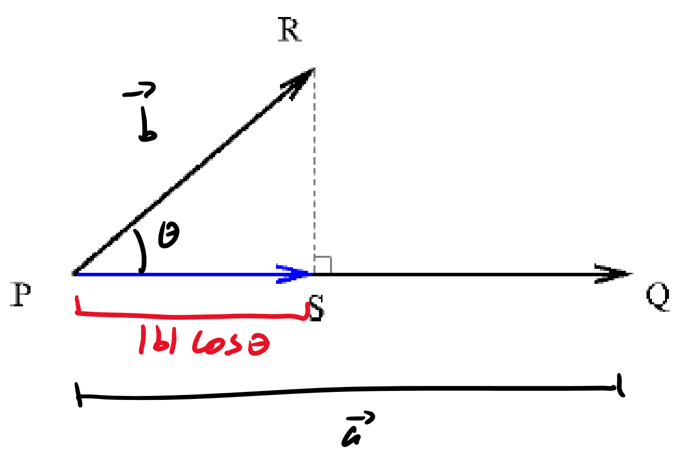
$$\theta = \arccos\left(\frac{22}{\sqrt{34}\sqrt{20}}\right)$$

$$\theta = 32.47^\circ$$

.567 Rad.

Scalar Projection and Vector Projection

The vector projection of $\mathbf{b} = \overrightarrow{PR}$ onto $\mathbf{a} = \overrightarrow{PQ}$, denoted as $\text{proj}_{\mathbf{a}}\mathbf{b}$, is the vector $\overrightarrow{PS}$.



scalar projection of b onto a

$$|\overrightarrow{PS}| = \text{comp}_a b = |b| \cos\theta$$

$$\text{comp}_a b = |b| \cos\theta \, \frac{|a|}{|a|}$$

$$= \frac{|a|\,|b| \cos\theta}{|a|}$$

$$\boxed{\text{comp}_a b = \frac{a \cdot b}{|a|}}$$

vector projection of b onto a

$$\overrightarrow{PS} = \text{proj}_a b = \frac{a \cdot b}{|a|} \cdot \frac{1}{|a|} a$$

$$\boxed{\text{proj}_a b = \frac{a \cdot b}{|a|^2} a}$$

Example: Find the scalar projection and the vector projection of b = ⟨3, 2⟩ onto a = ⟨4, 6⟩

$$\text{comp}_a b = \frac{a \cdot b}{|a|} = \frac{3(4) + 2(6)}{\sqrt{52}}$$

$$|a| = \sqrt{16 + 36} = \sqrt{52}$$

$$= \frac{12 + 12}{\sqrt{52}} = \frac{24}{\sqrt{52}}$$

$$\text{proj}_a b = \frac{a \cdot b}{|a|^2} a = \frac{24}{(\sqrt{52})^2} \langle 4, 6 \rangle = \frac{24}{52} \langle 4, 6 \rangle$$

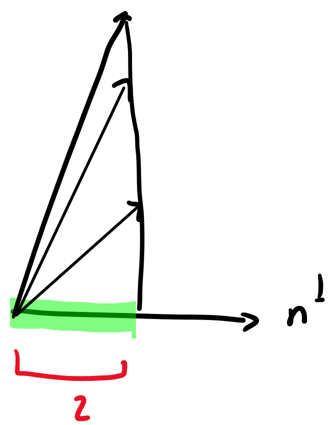
$$= \frac{6}{13} \langle 4, 6 \rangle = \boxed{\left\langle \frac{24}{13}, \frac{36}{13} \right\rangle}$$

Example: Find a vector $\mathbf{m}$ such that $\text{comp}_{\mathbf{n}^{\perp}}\mathbf{m} = 2$ and $\mathbf{n} = \langle 5, 12 \rangle$

$n^{\perp} = \langle -12, 5 \rangle$ $\qquad m = \langle x, y \rangle$

$|n^{\perp}| = \sqrt{144+25} = \sqrt{169} = 13$

$$\text{comp}_{n^{\perp}} m = \frac{m \cdot n^{\perp}}{|n^{\perp}|} = \frac{-12x + 5y}{13} = 2$$



$-12x + 5y = 26$

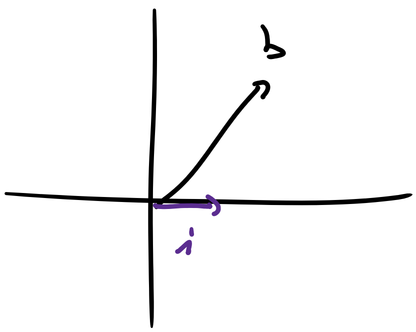
pick a # for x.

Let $x = 0 \longrightarrow y = \frac{26}{5}$

$m = \langle 0, \frac{26}{5} \rangle$

Example: Find the vector projection of $\mathbf{b} = \langle 3, 2\rangle$ onto $\mathbf{i} = \langle 1,0\rangle$

$$\text{proj}_i b = \frac{b \cdot i}{|i|^2}\, i = \frac{3(1) + 2(0)}{(1)^2}\, i = \frac{3}{1}\langle 1, 0\rangle = \langle 3, 0\rangle$$



$|a| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

Example: Find a unit vector in the same direction as the projection of $\mathbf{b} = \langle 5, 1\rangle$ onto $\mathbf{a} = \langle -1, 1\rangle$

$5(-1) + 1(1)$

$$\text{proj}_a b = \frac{b \cdot a}{|a|^2}\, a = \frac{-5+1}{(\sqrt{2})^2} \langle -1, 1\rangle$$

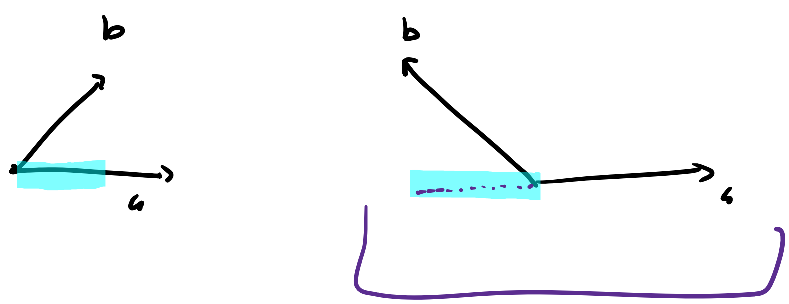
$$= \frac{-4}{2} \langle -1, 1\rangle = -2 \langle -1, 1\rangle$$

$$= \langle 2, -2\rangle$$

$$|\langle 2, -2\rangle| = \sqrt{(2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$$

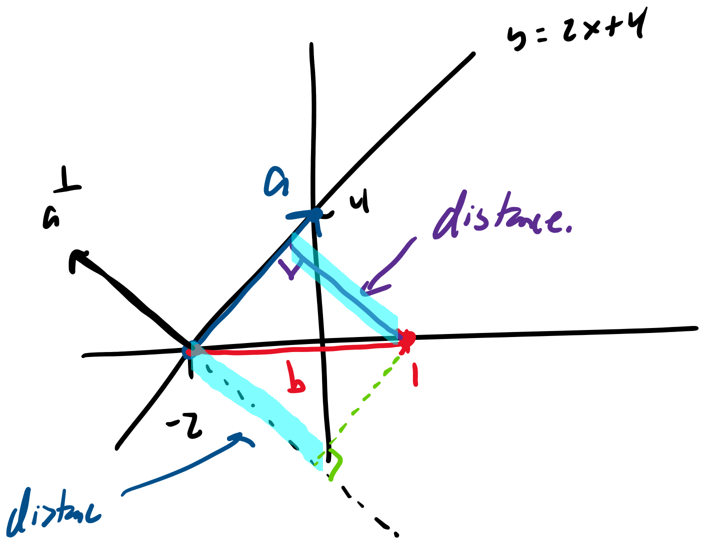
unit. $\frac{1}{\sqrt{8}} \langle 2, -2\rangle = \left\langle \frac{2}{\sqrt{8}}, \frac{-2}{\sqrt{8}} \right\rangle$

$$= \left\langle \frac{2}{2\sqrt{2}}, \frac{-2}{2\sqrt{2}} \right\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$



Pg 13 method 1

Example: Using vectors, find the distance from the point $(1, 0)$ to the line $y = 2x + 4$



1) need a slope vector
ie a vector on the line

$a = \langle 2, 4 \rangle$

2) need a vector b.

$b = \langle 3, 0 \rangle$

method 1

3) find $a^{\perp}$ $\qquad a^{\perp} = \langle -4, 2 \rangle$

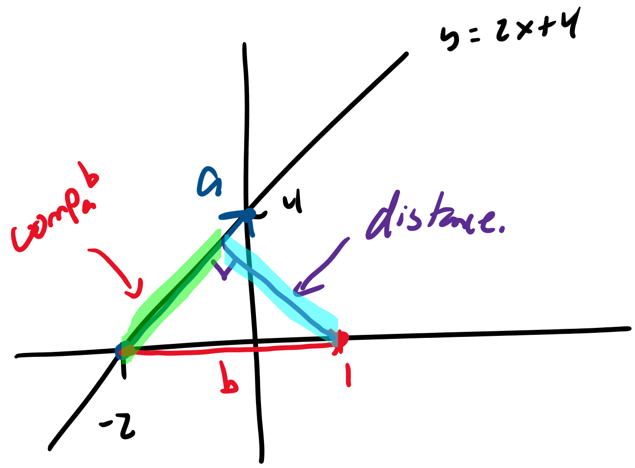
4) compute $\text{comp}_{a^{\perp}} b$

$$\text{comp}_{a^{\perp}} b = \frac{b \cdot a^{\perp}}{|a^{\perp}|} = \frac{-4(3) + 0(2)}{\sqrt{16+4}} = \frac{-12}{\sqrt{20}}$$

$$\text{Answer} = \frac{12}{\sqrt{20}}$$

Pg 13 method 2

Example: Using vectors, find the distance from the point $(1, 0)$ to the line $y = 2x + 4$



1) need a slope vector ie a vector on the line

$a = \langle 2, 4 \rangle$

2) need a vector b.

$b = \langle 3, 0 \rangle$

Method 2

3) compute $\text{comp}_a b = \dfrac{3(2) + 4(0)}{\sqrt{4+16}} = \dfrac{6}{\sqrt{20}}$

4) compute $|b| = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$

5) pythag. Thm.

$$|b|^2 = (\text{distance})^2 + (\text{comp}_a b)^2$$

$$9 = (\text{distance})^2 + \left(\frac{6}{\sqrt{20}}\right)^2$$

$$9 = (\text{distance})^2 + \frac{36}{20}$$

$$\frac{144}{20} = (\text{distance})^2$$

$$\sqrt{144} = \text{distance}$$

$$180 - 36 = 144$$

$$9 = \frac{180}{20}$$

$$\frac{180}{20} - \frac{36}{20} = \frac{144}{20}$$

12 :)

$\sqrt{\frac{144}{20}}$ = distance $\frac{12}{\sqrt{20}}$:)