Appendix J.3: Vector Functions

A vector function is a way to describe the a graph, or path of an object, using vectors. Vector functions are basically the same as parametric curves.

$x = t^4 + 2t + 5$

$y = (t^4 + 2t + 5)^2 + 1$

Example: Find a vector function that represents the function $y = x^2 + 1$.

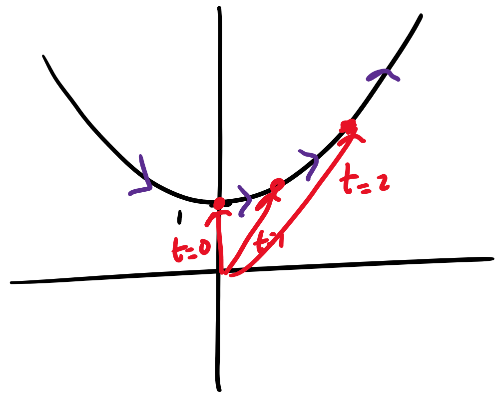
$r(t) = \langle x(t), y(t) \rangle$

parametric function:

$x = t \qquad y = t^2 + 1$

vector function

$r(t) = \langle t, t^2 + 1 \rangle$



$r(0) = \langle 0, 1 \rangle$

$r(1) = \langle 1, 2 \rangle$

$r(2) = \langle 2, 5 \rangle$

x y

$\langle t^2, t+2 \rangle$

Example: Use the vector function $\mathbf{r}(t) = t^2\mathbf{i} + (t+2)\mathbf{j}$ to answer the following.

A) Is the point $(4, 5)$ on the graph of $\mathbf{r}(t)$? Justify your answer.

$x = 4$ $\quad$ $y = 5$ $\quad$ $r(3) = \langle 9, 5 \rangle$ not the point

$t^2 = 4$ $\quad$ $t + 2 = 5$ $\quad$ $(4,5)$

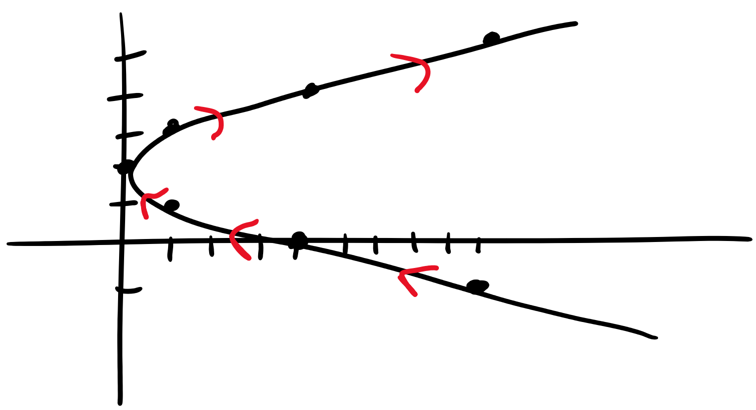
$t = 3$

So $\underline{\underline{\text{NO!}}}$

B) Sketch the graph of $\mathbf{r}(t)$. $= \langle t^2, t+2 \rangle$

points

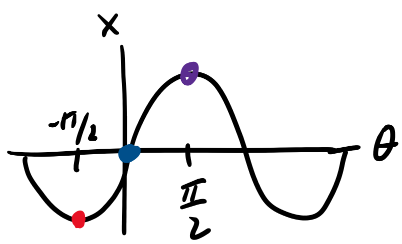
t	x	y
-3	9	-1
-2	4	0
-1	1	1
0	0	2
1	1	3
2	4	4
3	9	5



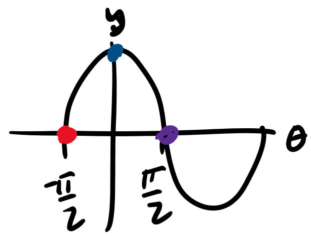
C) Find the Cartesian equation of $\mathbf{r}(t)$.

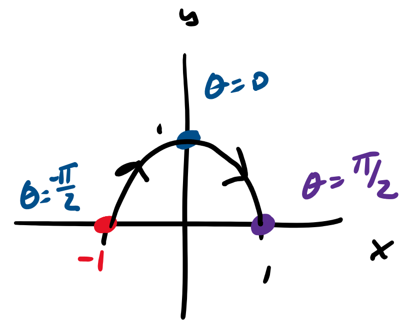
Example: Examine the vector function $\mathbf{r}(\theta) = \langle \overset{x}{\sin\theta}, \overset{y}{\cos\theta} \rangle$ where $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$x = \sin\theta$



$y = \cos\theta$





$\sin^2\theta + \cos^2\theta = 1$

Cartesian eq. $\longrightarrow$ $x^2 + y^2 = 1$ $\longrightarrow$ $y = \sqrt{1 - x^2}$

circle of Radius 1

$\sin(2\theta) = 2\sin\theta\cos\theta$

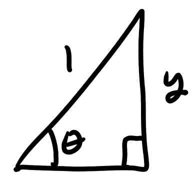
Example: Find the Cartesian equation of for parametric function.

$x = \sin(2\theta) = 2\sin\theta\cos\theta$
$y = \sin(\theta)$

$x = 2y\cos\theta$

$x = 2y\sqrt{1-y^2}$

$y = \sin\theta$



$\sqrt{1^2 - y^2} = \sqrt{1-y^2}$

$\cos\theta = \frac{\sqrt{1-y^2}}{1} = \sqrt{1-y^2}$

Example: Sketch the graph of the parametric curve. Give the Cartesian equation.

$x = 4\sin(t), \quad y = 3 + 4\cos(t)$

$t=0 \quad x=0 \quad y=7 \qquad t=\frac{\pi}{2} \quad x=4 \quad y=3$

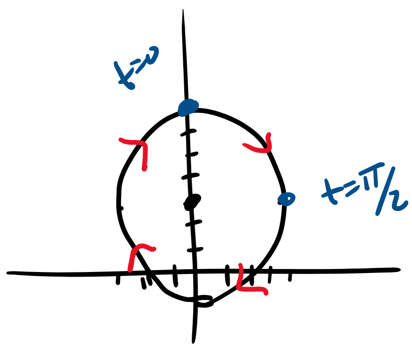
$\frac{x}{4} = \sin t \qquad \frac{y-3}{4} = \cos t$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{4}\right)^2 + \left(\frac{y-3}{4}\right)^2 = 1$

$\frac{x^2}{16} + \frac{(y-3)^2}{16} = 1$

$x^2 + (y-3)^2 = 16$



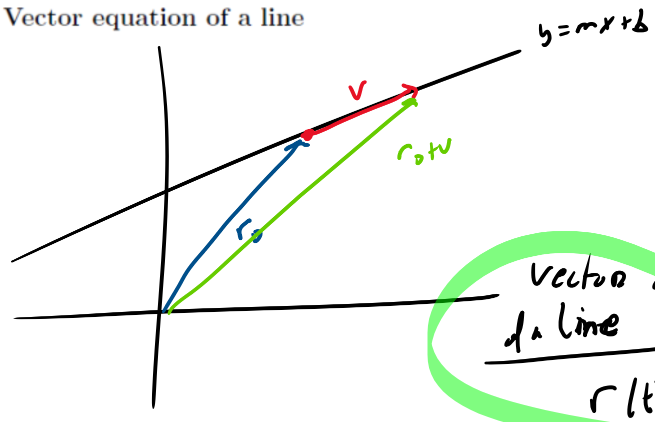
circle center $\underset{(0,3)}{(h,k)}$

Radius $= 4 = r$

$(x-h)^2 + (y-k)^2 = r^2$

Line.

Vector equation of a line



v = direction vector
"slope vector."

r_0 = vector to a point on the Line.

vector eq. of a line

$$r(t) = r_0 + tv$$

Example: Find a vector equation of the line through the points $A(1, 4)$ and $B(3, 9)$.

$v = \vec{AB} = \langle 2, 5\rangle$

$r_0 = \langle 1, 4\rangle$

$$r(t) = r_0 + tv$$
$$= \langle 1,4\rangle + t\langle 2,5\rangle$$
$$= \langle 1,4\rangle + \langle 2t, 5t\rangle$$
$$\boxed{r(t) = \langle 1+2t, 4+5t\rangle}$$

$v = \langle 4, 10\rangle$

$r_0 = \langle 3, 9\rangle$

$r(t) = \langle 3,9\rangle + t\langle 4,10\rangle$

$= \langle 3+4t, 9+10t\rangle$

parametric eq. as a comma separated list of equations

$x = 1+2t, \ y = 4+5t$

Example: Find a vector equation of the line $y = 7x + 5$

$m = \frac{7}{1}$

A $x=0$ $y=5$

B $x=1$ $y=12$

$\vec{AB} = \langle 1, 7 \rangle$

$V = \langle 1, 7 \rangle$

$r(t) = r_0 + tV$

$r(t) = \langle 0, 5 \rangle + t \langle 1, 7 \rangle$

$= \langle 0, 5 \rangle + \langle t, 7t \rangle$

$= \langle t, 5 + 7t \rangle$

$r = r_0 + tv$

Example: Are these lines parallel, orthogonal or neither? If they are not parallel, find the intersection point of these lines.

$\mathbf{L}_1(t) = (1+4t)\mathbf{i} + (9+16t)\mathbf{j} = \langle \overset{x}{1+4t}, \overset{y}{9+16t} \rangle$ $\quad V_1 = \langle 4, 16 \rangle$

$\mathbf{L}_2(s) = (-1+3s)\mathbf{i} + (25-6s)\mathbf{j} = \langle -1+3s, 25-6s \rangle$ $\quad V_2 = \langle 3, -6 \rangle$

V_1 and V_2 are not parallel.

so L_1 + L_2 are not parallel.

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is $V_1 \cdot V_2 = 0$?

$$V_1 \cdot V_2 = 4(3) + 16(-6) = 12 - 96 \neq 0$$

L_1 + L_2 are not perpendicular.

$L_1(t) = \langle \overset{x}{1+4t}, \overset{y}{9+16t} \rangle$ $\quad L_2(s) = \langle \overset{x}{-1+3s}, \overset{y}{25-6s} \rangle$

x values

$1+4t = -1+3s$

$4t = -2+3s$

$t = \dfrac{-2+3s}{4}$

$t = \frac{1}{2}$

y values

$9+16t = 25-6s$

$9+16\left(\dfrac{-2+3s}{4}\right) = 25-6s$

$9+4(-2+3s) = 25-6s$

$9-8+12s = 25-6s$

$1+12s = 25-6s$

$18s = 24$

$s = \frac{24}{18} = \frac{4}{3}$

$$s = \frac{24}{18} = \frac{4}{3}$$

$$L_2\left(\tfrac{4}{3}\right) = \left\langle -1 + 3\left(\tfrac{4}{3}\right), 25 - 6\left(\tfrac{4}{3}\right) \right\rangle$$
$$= \langle -1 + 4, 25 - 8 \rangle$$
$$= \langle 3, 17 \rangle$$

point $(3, 17)$