Section 2.3: Calculating Limits Using Limit Laws

Limit Laws Suppose that c is a constant and the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ are values ~~exists.~~ Then

1) $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

2) $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

3) $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$

4) $\lim\limits_{x\to a} f(x) * g(x) = \lim\limits_{x\to a} f(x) * \lim\limits_{x\to a} g(x)$

5) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim\limits_{x\to a} g(x) \neq 0$

6) $\lim\limits_{x\to a}[f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n$, where n is a positive integer

7) $\lim\limits_{x\to a} c = c$ 8) $\lim\limits_{x\to a} x = a$

9) $\lim\limits_{x\to a} x^n = a^n$ where n is a positive integer

10) $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and if n is even, then we assume that $a > 0$

11) $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, where n is a positive integer.

Example: Suppose $\underbrace{\lim_{x\to a} f(x) = 5}$, and $\underbrace{\lim_{x\to a} g(x) = 2}$, compute

$$\lim_{x\to a} \frac{2f(x) - 3g(x)}{(f(x))^2} = \frac{\lim\limits_{x\to a} (2f(x) - 3g(x))}{\lim (f(x))^2} = \frac{\lim\limits_{x\to a} 2f(x) - \lim\limits_{x\to a} 3g(x)}{\left(\lim\limits_{x\to a} f(x)\right)^2}$$

$$= \frac{2\boxed{\lim\limits_{x\to a} f(x)} - 3\boxed{\lim\limits_{x\to a} g(x)}}{\left(\boxed{\lim\limits_{x\to a} f(x)}\right)^2} = \frac{2(5) - 3(2)}{(5)^2}$$

$$= \frac{10 - 6}{25} = \frac{4}{25}$$

Example: Compute the following limits.

A) $\lim_{x\to 2} \left(4x^3 + 5\right) = \lim_{x\to 2} 4x^3 + \lim_{x\to 2} 5$

$= 4 \lim_{x\to 2} (x^3) + \lim_{x\to 2} 5$

$= 4 \left(\lim_{x\to 2} x\right)^3 + \lim_{x\to 2} 5$

$= 4(2)^3 + 5 \quad = 4(8) + 5 = 32 + 5 = 37$

B) $\lim_{x\to -3} \frac{x^2-9}{x+5}$ $= \frac{(-3)^2-9}{-3+5} = \frac{9-9}{2} = \frac{0}{2} = 0$

case. $\frac{0}{0}$ ← more Algebra needs to be done.

C) $\lim_{x\to -3} \frac{x^2-9}{x^2+x-6}$ $= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x-2)(x+3)} = \lim_{x\to -3} \frac{x-3}{x-2} = \frac{-3-3}{-3-2} = \frac{-6}{-5}$

$= \frac{6}{5}$

$f(x) = \frac{x^2-9}{x^2+x-6} = \frac{(x-3)(x+3)}{(x-2)(x+3)}$

domain is all Reals except $x=2$ + $x=-3$

$\frac{x-3}{x-2} = g(x)$

domain is all Reals except $x=2$

$\frac{0}{0}$

D) $\lim_{h\to 0} \frac{(4+h)^2-16}{h} = \lim_{h\to 0} \frac{16+8h+h^2-16}{h} = \lim_{h\to 0} \frac{8h+h^2}{h}$

$= \lim_{h\to 0} \frac{h(8+h)}{h} = \lim_{h\to 0} 8+h = 8+(0) = 8$

Example: Use the function $f(x)$ to answer these questions. $f(x) = \begin{cases} x^3-2x+4 & \text{if } x \le 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$

A) $\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^3-2x+4) = 1-2+4 = 3$

B) $\lim_{x\to 2} f(x) =$ DNE

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3-2x+4)$

$= 8-4+4 = 8$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x-2)$

$= 3(2)-2$

$= 6-2 = 4$

Example: Evaluate these limits.

A) $\lim_{x\to 3} \frac{|x-3|}{x-3}$ = DNE

$$|x-3| = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases}$$

$L = \lim_{x\to 3^+}$ $= \lim_{x\to 3^+}$

$$\lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{x\to 3^+} \frac{x-3}{x-3} = \lim_{x\to 3^+} 1 = 1$$

$$\lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-(x-3)}{x-3} = \lim_{x\to 3^-} -1 = -1$$

Case $\frac{0}{0}$

B) $\lim_{x\to 2} \frac{x^{-1} - .5}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2} = \lim_{x\to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x-2}$

$= \lim_{x\to 2} \frac{\frac{2-x}{2x}}{\frac{x-2}{1}} = \lim_{x\to 2} \frac{2-x}{2x} \cdot \frac{1}{x-2}$

$= \lim_{x\to 2} \frac{-(x-2)}{2x} \cdot \frac{1}{x-2} = \lim_{x\to 2} \frac{-1}{2x} = \frac{-1}{4}$

C) $\lim_{x\to 0} \dfrac{\left(\sqrt{3-x}-\sqrt{3}\right)}{x} \cdot \dfrac{\left(\sqrt{3-x}+\sqrt{3}\right)}{\left(\sqrt{3-x}+\sqrt{3}\right)} = \lim_{x\to 0} \dfrac{3-x-3}{x\left(\sqrt{3-x}+\sqrt{3}\right)}$

$= \lim_{x\to 0} \dfrac{-\cancel{x}}{\cancel{x}\cdot\left(\sqrt{3-x}+\sqrt{3}\right)} = \lim_{x\to 0} \dfrac{-1}{\sqrt{3-x}+\sqrt{3}}$

$= \dfrac{-1}{\sqrt{3-0}+\sqrt{3}} = \dfrac{-1}{2\sqrt{3}}$

Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ for all x in an open interval about a (except possibly at a) and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = L$

Example: Compute $\lim_{x \to 0^+} \sqrt{x} e^{sin(3/x)}$

$$-1 \leq \sin\left(\frac{3}{x}\right) \leq 1 \qquad x \neq 0$$

$$e^{-1} \leq e^{\sin \frac{3}{x}} \leq e^{1}$$

$$\sqrt{x}\, e^{-1} \leq \sqrt{x}\, e^{\sin \frac{3}{x}} \leq \sqrt{x}\, e^{1}$$

$$\lim_{x \to 0^+} \sqrt{x}\, e^{-1} = 0$$

$$\lim_{x \to 0^+} \sqrt{x}\, e^{1} = 0$$

by Squeeze Thrm

$$\lim_{x \to 0^+} \sqrt{x}\, e^{\sin\left(\frac{3}{x}\right)} = 0$$