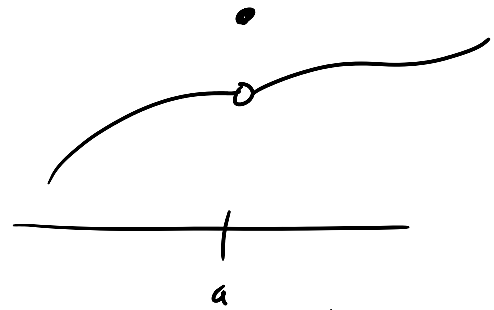
Section 2.5: Continuity

Definition: A function f is **continuous at a number** $x = a$ if $\lim_{x\to a} f(x) = f(a)$

1) $f(a)$ is defined as a value.

2) $\lim_{x\to a} f(x)$ is a value.

3) $\lim_{x\to a} f(x) = f(a)$



not cont. at $x=a$

Example: Is the function $f(x) = x^2 + 1$ continuous at $a = 3$, i.e. at $x = 3$?

$f(3) = 3^2 + 1 = 9 + 1 = 10$

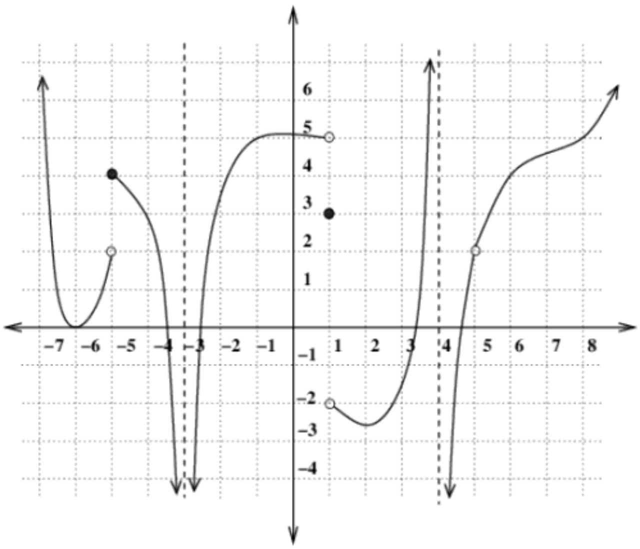
$\lim_{x\to 3} x^2 + 1 = 3^2 + 1 = 9 + 1 = 10$

equal so $f(x)$ is continuous at $x=3$

$f(x)$ is cont. for all Reals

polynomial are continuous

Example: Where is the function $f(x)$ discontinuous? Explain what type of discontinuity happens at that value of x.



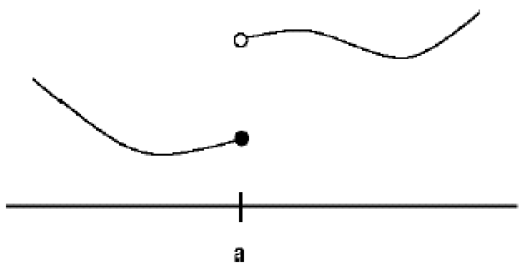
$x=-5$
$x=1$
jump discontinuities

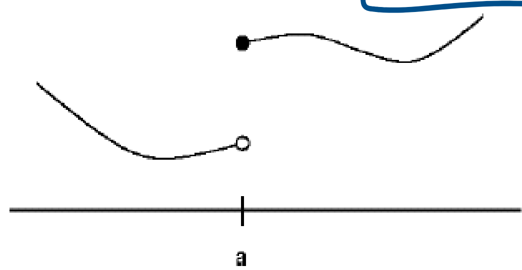
$x=-3$
$x=4$
vertical asymptotes

$x=5$
Removable discontinuity.

Definition: A function f is **continuous from the right at a number** a if $\lim_{x\to a^+} f(x) = f(a)$.

A function f is **continuous from the left at a number** a if $\lim_{x\to a^-} f(x) = f(a)$.





cont. from left.

conti from Right.

A function f is continuous on an interval if it is continuous at every number in the interval. At the endpoint of the interval we understand continuous to mean left or right continuity.

.1	4.1	9.1
.01	4.01	9.01
.001	4.001	9.001

Example: Discuss the continuity of the function $f(x) = \dfrac{x+5}{x-4}$.

Domain is all Reals except $x = 4$

$(-\infty, 4) \cup (4, \infty)$

$$\lim_{x \to 4^+} f(x) = +\infty$$

$$\lim_{x \to 4^-} f(x) = -\infty$$

$x = 4$ V.A. and $f(x)$ is not cont at $x = 4$

Example: Discuss the continuity of the function $f(x) = \dfrac{x^2+x-2}{x-1}$.

Domain is all Reals except $x=1$

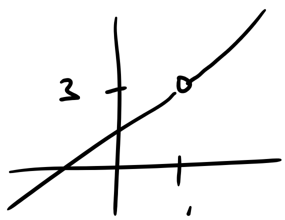
$\frac{0}{0}$

$$\lim_{x \to 1} \frac{x^2+x-2}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x-1}$$

$$= \lim_{x \to 1} x+2 = 3$$

not cont $x = 1$

It is a Removable discontinuity.



$$f(x) = \frac{x^2+x-2}{x-1} = \frac{(x-1)(x+2)}{x-1}$$

$$\underbrace{g = x+2}$$

This function fixes the Removable discontinuity.

Example: What would you define $f(1)$ to make the function continuous? i.e. Find the value of A so that $f(x)$ is continuous.

$$f(x) = \begin{cases} \dfrac{x^2+x-2}{x-1} & \text{if } x \neq 1 \\ A & \text{if } x = 1 \end{cases}$$

$f(1) = A$

$$\lim_{x \to 1} \frac{x^2+x-2}{x-1} = \lim_{x \to 1} \frac{(x+2)(x-1)}{x-1} = \lim_{x \to 1} (x+2) = 3$$

let $A = 3$

Example: Find the values where $f(x)$ is not continuous. Then classify the value(s) as a vertical asymptote or removable discontinuity.

$$f(x) = \frac{x^2 + 2x}{x^4 - 3x^3 - 10x^2} = \frac{x(x+2)}{x^2(x^2-3x-10)} = \frac{x(x+2)}{x^2(x-5)(x+2)}$$

Simplify

$$g(x) = \frac{1}{x(x-5)}$$

not cont at

$x=0 \leftarrow$ VA

$x=5 \leftarrow$ VA

$x=-2 \leftarrow$ Removable discontinuity

Example: Find the value(s) where $f(x)$ is not continuous.

$$f(x) = \begin{cases} 3x+1 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{cases}$$

The parts are continuous
Since they are polynomials.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x+1) = 3+1 = 4$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2$$

$\lim_{x \to 1} f(x)$ DNE
not cont. at $x=1$

Example: Find the value(s) where $f(x)$ is not continuous.

$$f(x) = \begin{cases} 3x & \text{if } x < 2 \\ x+4 & \text{if } x > 2 \end{cases}$$

Both parts are polynomials. (i.e. cont.)

$f(x)$ is not defined at $x=2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3x = 6$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x+4 = 6$$

$\rightarrow \lim_{x \to 2} f(x) = 6$

$f(x)$ not cont. at $x=2$

Removable discontinuity

Example: Find the value(s) of A that will make $g(x)$ a continuous function.

$$g(x) = \begin{cases} A^2 x & \text{if } x \le 1 \\ 3Ax - 2 & \text{if } x > 1 \end{cases}$$

g(x) is defined at x=1

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x)$$

$$\lim_{x \to 1^-} A^2 x = \lim_{x \to 1^+} (3Ax - 2)$$

$$A^2 = 3A - 2$$

$$A^2 - 3A + 2 = 0$$

$$(A-2)(A-1) = 0$$

$A = 1$ OR $A = 2$

Example: Find the value(s) where $f(x)$ is not continuous.

$$f(x) = \begin{cases} 3x^2 + 4x + 1 & \text{if } x \leq 2 \\ \dfrac{5x^2+1}{x-1} & \text{if } x > 2 \end{cases}$$

top part polynomial ✓
it is cont.

Bottom part.
x=1 is bad but not in domain so it is also good.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3x^2 + 4x + 1 = 12 + 8 + 1 = 21$$

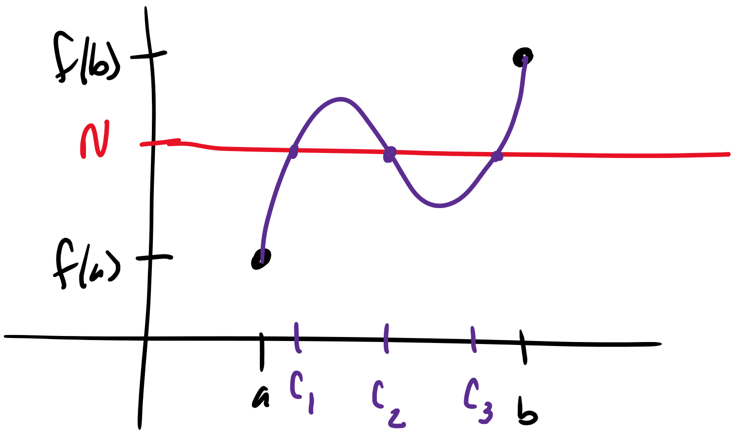
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{5x^2+1}{x-1} = \frac{5(4)+1}{2-1} = 21$$

$$\rightarrow \lim_{x \to 2} f(x) = 21$$

f(x) is continuous for All values

$a \le x \le b$

Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number such that N is strictly between $f(a)$ and $f(b)$. There there exist a number c with $a < c < b$ such that $f(c) = N$.



Example: Use the Intermediate Value Theorem to show that there is areal number a such that $f(a) = 12$.

$f(x) = -x^4 + 3x^3 + 5$

f(x) is a polynomial & thus it is continuous.

$f(0) = 5$

$f(1) = -1 + 3 + 5 = 7$

$f(2) = -16 + 24 + 5 = 13$

By The I.V.T. there is a # c such that $1 < c < 2$ and $f(c) = 12$

Example: Show that $f(x) = x^4 - 5x^2$ and $g(x) = 2x^3 - 4x + 6$ intersect between $x = 3$ and $x = 4$.

$f(x) = g(x)$

$f(x) - g(x) = 0$

Let $J(x) = f(x) - g(x)$

$= x^4 - 5x^2 - (2x^3 - 4x + 6)$

$J(x) = x^4 - 5x^2 - 2x^3 + 4x - 6$ $\longleftarrow$ $J(x)$ is continuous.
it is a polynomial

$J(3) = -12$

$J(4) = 58$

By I.V.T. There is a # c such that $3 < c < 4$ and $J(c) = 0$.

but $J = f(x) - g(x)$ so.

$f(c) - g(c) = 0$

$\underline{f(c) = g(c)}$

Example: A student did the following work on a question on an exam. The student showed that $f(1) = 1$ and $f(-1) = -1$ for the given function and then claimed by the Intermediate Value Theorem that there was some number c with $-1 < c < 1$ such that $f(c) = 0$. Did the student receive full credit on this problem?

NO!

statement can imply this function

$$f(x) = \frac{1}{x}$$