Section 3.10: Linear approximation and Differentials

Linear Approximation

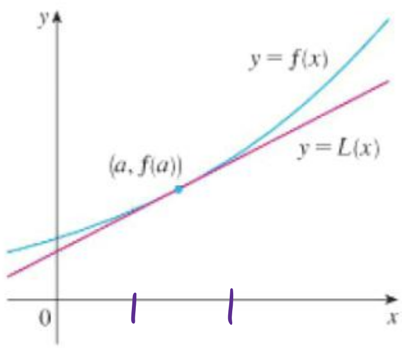
Definition The formula $L(x) = f(a) + f'(a)(x - a)$ is called the **linear approximation** or **linearization** of $f(x)$ at $x = a$.

Tangent line eq. at $x=a$

$y = f(a) + f'(a)(x-a)$

$y - f(a) = f'(a)(x-a)$

$y = mx+b$.



$y = e^x$

$y' = e^x$

$m = e^0 = 1$

Example: Use $y = e^x$ to answer these questions.

A) Find the linearization at $a = 0$. (tangent at $x = 0$)

point $(0, e^0) = (0, 1)$

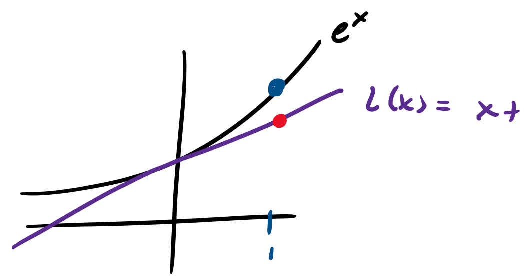
Tangent Line $\quad y - 1 = 1(x - 0)$

$y - 1 = x$

$y = x + 1$

$L(x) = x + 1$

B) Use the linearization to approximate e^1 and $\dfrac{1}{e^{.25}}$



$e^1 \approx L(1) = 1 + 1 = 2$

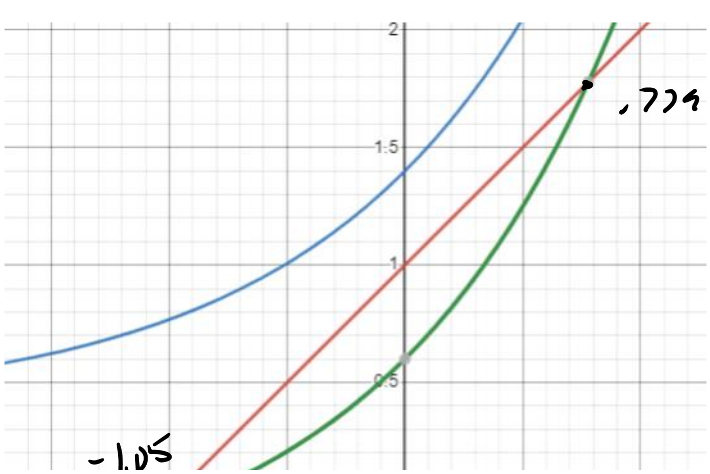
$\dfrac{1}{e^{.25}} = e^{-.25} \approx L(-.25) = -.25 + 1 = .75$

C) Find the values of x where the approximation is accurate to within 0.4.

$|L(x) - f(x)| < .4$

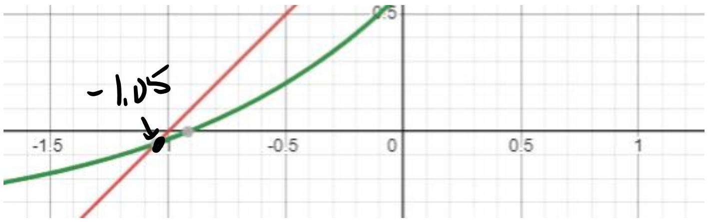
$-.4 < L(x) - e^x < .4$

$-.4 + e^x < L(x) < .4 + e^x$



$-1.05 < x < .779$

accuracy within .4



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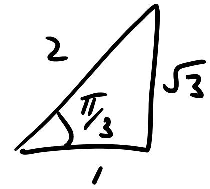
Example: Find the linearization of $y = \cos(x)$ at $a = 60^o$. Use it to estimate $\cos(61^o)$ and $\cos(59^o)$.

$a = \frac{\pi}{3} = \frac{60\pi}{180}$



$L(x) = f(a) + f'(a)(x-a)$

$y = \cos(x)$

$y' = -\sin(x)$

$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$f'\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = \frac{-\sqrt{3}}{2}$

$$L(x) = \frac{1}{2} + \frac{-\sqrt{3}}{2}\left(x - \frac{60\pi}{180}\right)$$

$$L(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$\cos(61) = \cos\left(\frac{61\pi}{180}\right) \approx L\left(\frac{61\pi}{180}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{61\pi}{180} - \frac{60\pi}{180}\right)$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) = .48488500 53$$

$$\cos(59) = \cos\left(\frac{59\pi}{180}\right) \approx L\left(\frac{59\pi}{180}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{-\pi}{180}\right)$$

$$= .5151149947$$

Example: Use $y = \sqrt{x+7}$ to answer these questions.

$y(2) = \sqrt{9} = 3$

$y' = \frac{1}{2}(x+7)^{-1/2}(1) = \frac{1}{2\sqrt{x+7}}$

$y'(2) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$

A) Find the linearization at $a = 2$

$\sqrt{x+7} \approx L(x) = f(2) + f'(2)(x-2)$

$L(x) = 3 + \frac{1}{6}(x-2)$

$L(x) = 3 + \frac{x}{6} - \frac{2}{6} = 3 + \frac{x}{6} - \frac{1}{3}$

$L(x) = \frac{8}{3} + \frac{x}{6}$

B) Evaluate $\sqrt{9.06}$ and $\sqrt{11}$

$\sqrt{9.06} = \sqrt{2.06+7} \approx L(2.06) = 3 + \frac{1}{6}(2.06-2) = 3.01$

$\sqrt{11} = \sqrt{4+7} \approx L(4) = 3 + \frac{1}{6}(4-2) = \frac{10}{3}$

C) Find the values of x where the approximation is accurate to within 0.5.

Definition let $y = f(x)$, where f is a differentiable function. Then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation $dy = f'(x)dx$.

$$\frac{dy}{dx} = f'(x)$$

$$dy = f'(x)\,dx$$

Example: Find dy and evaluate dy for the values of $x = 2$ and $dx = 0.3$.

$y = x^3 + 2x + 7$

$$\frac{dy}{dx} = 3x^2 + 2$$

$$dy = (3x^2+2)\,dx$$

$$dy = (3(2)^2+2)(.3)$$
$$= 14(.3)$$
$$dy = 4.2$$

$x=2 \quad y = 2^3 + 2(2) + 7 = 8 + 4 + 7 = 19$

dy is the approximate change in the y variable from $x=2$ to $x = 2 + dx = 2.3$

Actual change is $\Delta y = y(2.3) - y(2) = 23.767 - 19 = 4.767$

Actual change.

Example: Find dy and evaluate dy for the values of $x = 1$ and $dx = 0.4$.

$y = \sqrt{x^2 + 3}$

$$\frac{dy}{dx} = \frac{1}{2}(x^2+3)^{-1/2} \cdot 2x$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+3}}$$

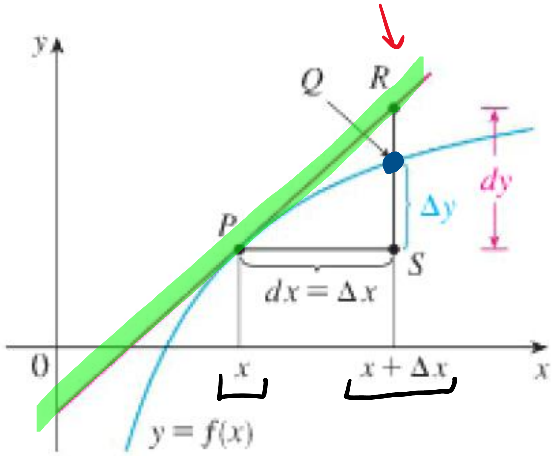
$$dy = \frac{x}{\sqrt{x^2+3}}\,dx$$

$$dy = \frac{1}{\sqrt{1^2+3}}(.4) = \frac{1}{\sqrt{4}}(.4)$$

$$dy = \frac{1}{2}(.4) = .2$$

$$\boxed{\sqrt{(1.4)^2+3} \approx f(1) + dy = 2 + .2 = 2.2}$$

$m = \frac{dy}{dx}$



we have $dx = \Delta x$

$$f(x+\Delta x) = f(x) + \Delta y$$

$$f(x+\Delta x) \approx f(x) + dy$$

what is dy?

$dy = f'(x)\,dx$

if $x = a$ and $\Delta x = dx$

Then $f(a+\Delta x) = f(a) + \Delta y \approx f(a) + dy = f(a) + f'(a)\,dx$

$= f(a) + f'(a)(x-a)$

$\Delta x = dx = (x - a)$

Example: Use differentials to estimate $\sqrt[4]{16.1}$.

$y = \sqrt[4]{x}$

$x = a = 16 \qquad dx = .1$

$f(16) = y(16) = \sqrt[4]{16} = 2$

$\frac{dy}{dx} = \frac{1}{4} x^{-3/4}$

$dy = \frac{1}{4x^{3/4}} dx$

$dy = \frac{1}{4 \cdot 16^{3/4}} (.1) = \frac{.1}{4(8)}$

$dy = \frac{.1}{32} = \frac{1}{320}$

$\sqrt[4]{16.1} \approx f(16) + dy = 2 + \frac{1}{320} = \frac{641}{320}$

Example: The edge of a cube is measured to be 20 inches with a maximum error of 0.1 inches. What is the maximum error in the volume? What is the relative error? What is the percentage error?

dV

$V = x^3$ $\qquad$ $x = 20$ $\qquad$ $dx = .1$

$\frac{dV}{dx} = 3x^2$

$dV = 3x^2\,dx$

$dV = 3(20)^2(.1) = 120$ cubic inches

max error in Vol.

Relative error $\frac{dV}{V} = \frac{120}{(20)^3} = \frac{120}{8000}$

$= .015$

Percent error

1.5%

Example: Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with radius of 25 m.

Sphere $V = \frac{4}{3}\pi r^3$

hemisphere $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$

$V = \frac{2}{3}\pi r^3$

$r = 2500\text{ cm}$

$dr = .05$

$\frac{dv}{dr} = 2\pi r^2$

$dv = 2\pi r^2\, dr$

$dv = 2\pi(2500)^2(.05)$

$= 625000\pi = \underbrace{1963495.408 \text{ cm}^3}_{\text{Amount of paint.}}$