Section 3.2: Product and Quotient Rules

Product Rule If $y = \underbrace{f(x)g(x)}$ and both $f'(x)$ and $g'(x)$ exists then

$y' = f'(x)g(x) + f(x)g'(x)$

$y = fg$

$y' = f'g + fg'$

$= f'g + g'f$

$y = fgh$

$y' = f'gh + fg'h + fgh'$

Example: Find the derivative of $y = \underbrace{(x^6+7)}_{f}\underbrace{(x^2+x)}_{g}$

$y = x^8 + x^7 + 7x^2 + 7x$

$y' = 8x^7 + 7x^6 + 14x + 7$

$f' = 6x^5$

$g' = 2x+1$

$y = fg$

$y' = f'g + fg'$

$= 6x^5 \cdot (x^2+x) + (x^6+7) \cdot (2x+1)$

Example: Find f'' for $f(x) = \underbrace{(x^6+7)}_{f}\underbrace{e^x}_{g}$

$y' = f'g + fg'$

$f'(x) = 6x^5 \cdot e^x + (x^6+7)e^x$

$= (6x^5 + x^6 + 7)e^x$

$f'' = (30x^4 + 6x^5)e^x + (6x^5 + x^6 + 7) \cdot e^x$

Quotient Rule If $y = \frac{f(x)}{g(x)}$ and both $f'(x)$ and $g'(x)$ exists then

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$y = \frac{Hi}{Lo}$$

$$y' = \frac{gf' - fg'}{(g)^2}$$

$$y' = \frac{L\,dH - H\,dL}{L^2}$$

$$y' = \frac{Lo\,dHi - Hi\,dLo}{(Lo)^2}$$

Example: Find the derivatives of these functions.

A) $y = \frac{1 - x^2}{1 + x^2}$

$$y' = \frac{L\,dH - H\,dL}{L^2}$$

$$y' = \frac{\overset{L}{(1+x^2)}\overset{dH}{(-2x)} - \overset{H}{(1-x^2)}\overset{dL}{2x}}{(1+x^2)^2}$$

$$\frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2}$$

$$= \frac{-4x}{(1+x^2)^2}$$

B) $y = \dfrac{5}{m^6 + 2}$

$$y' = \frac{\overset{L}{(m^6+2)}\,\overset{DH}{(0)} - \overset{H}{5} \cdot \overset{dL}{6m^5}}{(m^6+2)^2} = \frac{-30m^5}{(m^6+2)^2}$$

$dH = 6x^2 + 6x$

$dL = 2x + 2$

Example: Find y'' for $y = \dfrac{x^3}{x+1}$

$$y' = \frac{\overset{L}{(x+1)} \cdot \overset{DH}{3x^2} - \overset{H}{x^3}\,\overset{dL}{(1)}}{(x+1)^2} = \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \frac{2x^3 + 3x^2}{x^2 + 2x + 1} \begin{matrix} H \\ \\ L \end{matrix}$$

$$y'' = \frac{\overset{L}{(x^2+2x+1)}\overset{DH}{(6x^2 + 6x)} - \overset{H}{(2x^3 + 3x^2)}\,\overset{DL}{(2x+2)}}{(x^2+2x+1)^2}$$

$$\frac{LdH - HdL}{L^2}$$

$$y - f(A) = f'(A)(x - A)$$

Example: Find the equation of the tangent line at $x = 1$ $f(x) = \dfrac{x^2e^x}{x^5+3}$

$$f(1) = \frac{1^2e^1}{1^5+3} = \frac{1e^1}{4} = \frac{1}{4}e^1$$

$$f'(x) = \frac{(x^5+3)\cdot\frac{d}{dx}x^2e^x - x^2e^x\cdot\frac{d}{dx}(x^5+3)}{(x^5+3)^2}$$

$$f'(x) = \frac{(x^5+3)\left[2xe^x + x^2e^x\right] - x^2e^x\cdot 5x^4}{(x^5+3)^2}$$

$$f'(1) = \frac{4\left(2e^1 + 1e^1\right) - 1e^1\cdot 5}{4^2} = \frac{4(3e^1) - 5e^1}{16}$$

$$f'(1) = \frac{12e^1 - 5e^1}{16} = \frac{7e^1}{16} = m_{tan}$$

$$y - \frac{1}{4}e^1 = \frac{7}{16}e^1(x-1)$$

Example: The functions f and g that satisfy the properties as shown in the table. Find the indicated quantity.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−3	3	5
1	2	9	7	11
2	−5	0	2	10
3	4	−1	−4	8

A) $H'(3)$ if $H(x) = (x^3 + 2)g(x)$

$$H'(x) = 3x^2 g(x) + (x^3+2) \cdot g'(x)$$

$$H'(3) = 3(3)^2 g(3) + (3^3+2)\, g'(3)$$

$$= 27 g(3) + 29\, g'(3) = 27(-4) + 29(8) = 124$$

B) $\left.\dfrac{d}{dx}\left(\dfrac{x^3}{f(x)}\right)\right|_{x=1}$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	−3	3	5
1	2	9	7	11
2	−5	0	2	10
3	4	−1	−4	8

find $H'(1)$ if $H(x) = \dfrac{x^3}{f(x)}$

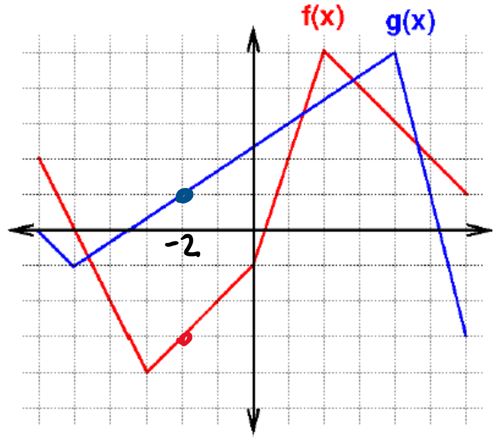
$$H'(x) = \frac{f(x) \cdot 3x^2 - x^3 \cdot f'(x)}{(f(x))^2}$$

$$H'(1) = \frac{f(1) \cdot 3 - 1 \cdot f'(1)}{(f(1))^2} = \frac{2(3) - 1 \cdot 9}{(2)^2} = \frac{6-9}{4} = \frac{-3}{4}$$

$f(-2) = -3$ $\quad g'(-2) = \frac{2}{3}$ $\quad f'(-2) = 1$

$g(-2) = 1$

Example Use the graph for the following.



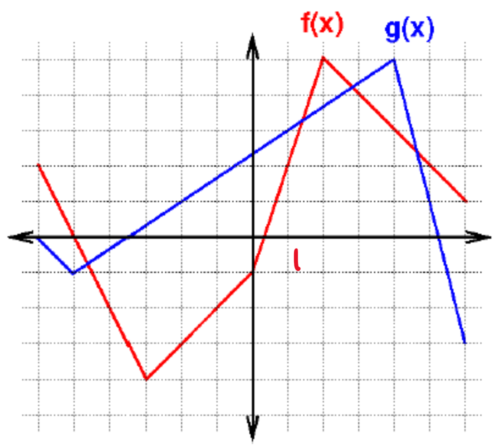
Find $H'(-2)$ if $H(x) = f(x)g(x)$

$H'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

$H'(-2) = f'(-2)\,g(-2) + f(-2)\,g'(-2)$

$= 1\,(1) + (-3)\left(\frac{2}{3}\right) = 1 - 2 = -1$

B) Find $R'(1)$ if $R(x) = \dfrac{x^2+2}{f(x)}$



$$R'(x) = \frac{f(x)\cdot 2x - (x^2+2)\,f'(x)}{(f(x))^2}$$

$$R'(1) = \frac{f(1)\cdot 2 - 3f'(1)}{(f(1))^2}$$

$f(1) = 2$ $\quad f'(1) = 3$

$$= \frac{2(2) - 3(3)}{2^2} = \frac{4-9}{4} = \frac{-5}{4}$$