Section 3.4: Chain Rule

Example: Find the derivative of $y = (x^3 + 3x^2 + 1)^5$

$u = x^3 + 3x^2 + 1$ $\quad$ $y = u^5$

$\frac{du}{dx} = 3x^2 + 6x$ $\quad$ $\frac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 5u^4 \cdot (3x^2 + 6x) = 5(x^3 + 3x^2 + 1)^4 (3x^2 + 6x)$$

The Chain Rule: If derivatives $g'(x)$ and $f'(x)$ both exist, and $J(x) = f(g(x))$ then

$J'(x) =$ $f'(g(x)) \cdot g'(x)$

In Leibniz notation: $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Example: Find the following derivatives.

A) $y = (x^5 + 7x^2 + 6)^8$

$$y' = 8(x^5 + 7x^2 + 6)^7 \cdot (5x^4 + 14x)$$

B) $y = e^{x^3+2x+1}$

$$y' = (3x^2 + 2)e^{x^3+2x+1}$$

C) $y = e^{\sec(5x)}$

$$y' = e^{\sec(5x)} \cdot \sec(5x)\tan(5x) \cdot 5$$

$y = e^x$

$y' = e^x$

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$y = e^{f(x)}$

$y' = e^{f(x)} \cdot f'(x)$

$y' = f'(x)\, e^{f(x)}$

D) $y = a^{f(x)}$ where a is a number.

is just a #

$$y = e^{\ln a^{f(x)}} = e^{f(x)\cdot \underbrace{\ln(a)}}$$

$$y' = e^{f(x)\ln(a)} \cdot f'(x)\ln(a) = f'(x)\ln(a)\, e^{\ln(a)^{f(x)}} = f'(x)\ln(a)\, a^{f(x)}$$

$$y = a^{f(x)} \qquad y' = f'(x)\, a^{f(x)} \ln(a)$$

E) $y = 2^{x^2}$

$$y' = 2x \cdot 2^{x^2} \ln(2)$$

$\ln(2) = \log_e(2)$

$\log x = \log_{10} x$

F) $y = \tan^3(5x) = \left(\tan(5x)\right)^3$

$$y' = 3\left(\tan(5x)\right)^2 \cdot \sec^2(5x) \cdot 5$$

$$= 15\left(\tan(5x)\right)^2\left(\sec(5x)\right)^2$$

G) $y = 5^{x^2}\sqrt[3]{x^2+7} = \underbrace{5^{x^2}}_{f} \cdot \underbrace{(x^2+7)^{1/3}}_{g}$

$$y' = \underbrace{2x \cdot 5^{x^2} \cdot \ln(5)}_{f'} \cdot \underbrace{(x^2+7)^{1/3}}_{g} + \underbrace{5^{x^2}}_{f} \underbrace{\tfrac{1}{3}(x^2+7)^{-2/3} \cdot 2x}_{g'}$$

H) $y = \dfrac{x^4+7}{(x^2+1)^3}$

$y = (x^4+7)(x^2+1)^{-3}$

$$y' = \frac{(x^2+1)^3 \cdot 4x^3 - (x^4+7)\,3(x^2+1)^2 \cdot 2x}{(x^2+1)^6}$$

$((x^2+1)^3)^2 =$
$= (x^2+1)^3(x^2+1)^3$
$= (x^2+1)^6$

Lets simplify

$$y' = \frac{(x^2+1)^2 \cdot 2x\left[(x^2+1)2x^2 - (x^4+7)\cdot 3\right]}{(x^2+1)^6}$$

$$= \frac{2x\left[2x^4+2x^2-3x^4-21\right]}{(x^2+1)^4} = \frac{2x(-x^4+2x^2-21)}{(x^2+1)^4}$$

I) $y = \dfrac{2}{(7x^2+5)^3} \longrightarrow y = 2(7x^2+5)^{-3}$

$$y' = \frac{(7x^2+5)^3 \cdot (0) - 2 \cdot 3(7x^2+5)^2 \cdot 14x}{(7x^2+5)^6}$$

$$y' = -6(7x^2+5)^{-4} \cdot 14x$$

$$= \frac{-84x}{(7x^2+5)^4}$$

J) $y = \left(\dfrac{\sin(x)}{x^4+3}\right)^5$

$$y' = 5\left(\frac{\sin(x)}{x^4+3}\right)^4 \cdot \left[\frac{(x^4+3)\cos(x) - \sin(x) \cdot 4x^3}{(x^4+3)^2}\right]$$

Example: Find the 25th derivative of $y = cos(7x)$

$y' = -\sin(7x) \cdot 7$

$y'' = -\cos(7x) \cdot 7 \cdot 7$

$= -7^2 \cos(7x)$

$y''' = 7^3 \sin(7x)$

$y^{(4)} = 7^4 \cos(7x)$

$$4 \overline{)25} = 6, \quad 25 - 24 = 1$$

Remainder 1

$y^{(25)} = -7^{25} \sin(7x)$

Example: Find the values of x where the tangent line is horizontal for

$y = \underbrace{(x^2-4)^3}_{f}\underbrace{e^{x^2}}_{g}$ $\hookrightarrow$ $m_{tan} = 0$

$$y' = \underbrace{3(x^2-4)^2\, 2x}_{f'} \cdot \underbrace{e^{x^2}}_{g} + \underbrace{(x^2-4)^3}_{f}\,\underbrace{2x e^{x^2}}_{g'}$$

$$y' = (x^2-4)^2\, 2x\, e^{x^2}\left[3 + (x^2-4)\right]$$

$$= (x^2-4)^2\, 2x\, e^{x^2}\,(3 + x^2 - 4)$$

$$y' = (x^2-4)^2\, 2x\, e^{x^2}\,(x^2-1)$$

$$0 = (x^2-4)^2\, 2x\, e^{x^2}\,(x^2-1)$$

$0 = x^2-4$ or $0 = 2x$ or $0 = e^{x^2}$ or $0 = x^2-1$

$4 = x^2$

$x = \pm\sqrt{4}$

$x = \pm 2$

$x = 0$

$\underbrace{0 = e^{x^2}}_{\text{no solution}}$

$X = \pm 1$

$$X = 0, 1, 2, -1, -2$$

Example: Find the 5th derivative of $y = xe^{-x}$

$y' = 1e^{-x} + x \cdot (-1) e^{-x}$

$y' = (1-x)e^{-x}$

$y'' = (-1)e^{-x} + (1-x)(-1)e^{-x}$

$= (-1 \; -1 + x)e^{-x}$

$y'' = (-2+x)e^{-x}$

$y''' = 1e^{-x} + (-2+x)(-1)e^{-x}$

$= (1 \; +2 - x)e^{-x}$

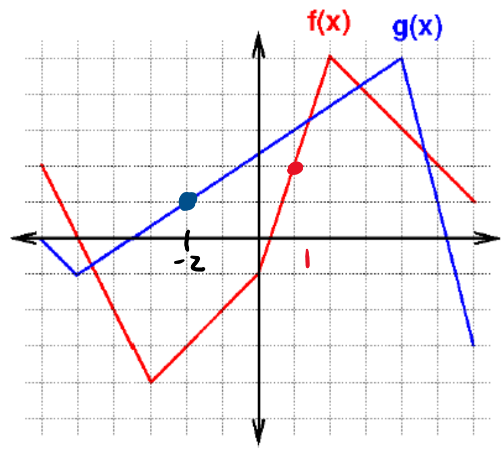
$y''' = (3-x)e^{-x}$

$y^{(4)} = (-1)e^{-x} + (3-x)(-1)e^{-x}$

$= (-4+x)e^{-x}$

$y^{(5)} = (5-x)e^{-x}$

Example Use the graph for the following.



A) Find $H'(-2)$ if $H(x) = f(g(x))$

$g(-2) = 1$ $\quad g'(-2) = \frac{2}{3}$

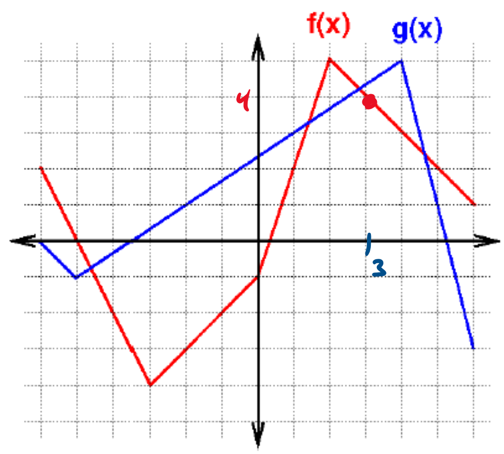
$f'(1) = 3$

$$H'(x) = f'(g(x)) \cdot g'(x)$$

$$H'(-2) = f'(g(-2)) \cdot g'(-2)$$

$$= f'(1) \cdot \frac{2}{3} = 3\left(\frac{2}{3}\right) = 2$$

B) Find $R'(1)$ if $R(x) = (x^2 + 2)f(3x)$



$f(3) = 4$ $\quad f'(3) = -1$

$$R'(x) = 2x\, f(3x) + (x^2+2) \cdot f'(3x) \cdot 3$$

$$R'(1) = 2\, f(3) + 3\, f'(3) \cdot 3$$

$$= 2(4) + 9(-1)$$

$$= -1$$