

Section 3.8: Exponential Growth and Decay

Exponential growth and decay is when the rate of change of a function is proportional to itself. In other words, an exponential growth/decay function will satisfy the differential equation

$$y' = ky$$

The variable k is the relative growth rate.

$$y = y_0 e^{kx}$$

$$y' = y_0 k e^{kx} = k y_0 e^{kx}$$

$$y' = ky$$

differential eq. (308)

$$y' + 3y = 7$$

$$y' + 3y = 7x + 2$$

$$y'' + 2y' + y = 8$$

Example: A bacteria culture starts with 1000 critters and 3 hours later it has 3000 critters. Assuming that the growth of the culture is proportional to its size, how many bacteria are present after 9 hours? How long does it take the population to double?

$$y' = ky$$

$$(0, 1000) \leftarrow$$

$$y = y_0 e^{kx} \quad \text{model.}$$

$$(3, 3000)$$

$$1000 = y_0 e^{k(0)} = y_0$$

$$y = 1000 e^{kx}$$

$$3000 = 1000 e^{k(3)}$$

$$3 = e^{3k}$$

$$\ln(3) = \ln e^{3k} = 3k$$

$$\frac{1}{3} \ln(3) = k$$

$$\text{mistake } y = 1000 e^{\frac{x}{3} \ln(3)}$$

A) after 9 hrs

$$y(9) = 1000 e^{\frac{9}{3} \ln(3)}$$

$$= 1000 e^{3 \ln(3)}$$

$$= 1000 e^{\ln(3^3)} = 1000 \cdot 3^3$$

$$= 27000$$

B) double)

$$2000 = 1000 e^{\frac{x}{3} \ln(3)}$$

$$2 = e^{\frac{x}{3} \ln(3)}$$

$$\ln 2 = \frac{x}{3} \ln(3)$$

$$3 \ln(2) = x \ln(3)$$

$$x = \frac{3 \ln(2)}{\ln(3)} \approx 1.892789 \text{ hrs}$$

Example: A bacteria culture starts with 1000 critters and 3 hours later it has 3000 critters. Assuming that the growth of the culture is proportional to its size, how many bacteria are present after 9 hours? How long does it take the population to double?

$$(0, 1000)$$

$$(3, 3000)$$

$$\rightarrow y_0 = 1000$$

$$y = y_0 a^x$$

$$y = 1000 a^x$$

$$3000 = 1000 a^3$$

$$3 = a^3$$

$$a = 3^{1/3}$$

model

$$y = 1000 \cdot 3^{x/3}$$

A) $y(9) = 1000 \cdot 3^{9/3}$

$$= 1000 \cdot 3^3$$

$$= 27000$$

B)

$$2000 = 1000 \cdot 3^{x/3}$$

$$2 = 3^{x/3}$$

$$\ln(2) = \ln 3^{x/3}$$

$$\ln(2) = \frac{x}{3} \ln(3)$$

$$x = \frac{3 \ln(2)}{\ln(3)}$$

$$y = y_0 e^{kx}$$

$$y = y_0 a^x$$

$$e^k = a$$

$$k = \ln(a)$$

Example: Three hours after a cell culture is started it has 278 cells in it. Four hours later the culture has 432 cells. Assuming that the growth of the population is proportional to the size, find a formula that would express the size of the culture at time x , where x is the number of hours since the start of the culture.

$$\rightarrow y' = ky$$

$$y = y_0 a^x$$

$$(3, 278)$$

$$(7, 432)$$

$$278 = y_0 a^3$$

$$432 = y_0 a^7$$

$$y_0 = \frac{278}{a^3}$$

$$432 = \frac{278}{a^3} a^7$$

$$432 = 278 a^4$$

$$\frac{432}{278} = a^4$$

$$a = \left(\frac{432}{278}\right)^{\frac{1}{4}}$$

$$y_0 = \frac{278}{\left(\frac{432}{278}\right)^{\frac{3}{4}}} \approx 199.74$$

$$y = 199.74 \left(\frac{432}{278}\right)^{\frac{x}{4}}$$

$$y = 199.74 e^{\frac{x}{4} \ln\left(\frac{432}{278}\right)}$$

Example: Strontium-90 has a half-life of about 29 years.

A) Find a function that would give the amount of strontium-90 remaining at time t for a given sample.

B) How much will be left after 14 years?

C) How long until only $1/8$ of the sample remains?

D) How long until 60% of the sample is gone?

$$(0, 100\%)$$

$$(29, 50)$$

$$y = y_0 e^{kx}$$

$$y = 100 e^{kx}$$

$$50 = 100 e^{k \cdot 29}$$

$$\frac{1}{2} = e^{k \cdot 29}$$

$$\ln\left(\frac{1}{2}\right) = 29k$$

$$k = \frac{\ln(0.5)}{29}$$

$$d) \quad y_0 = 100 e^{kx}$$

$$.4 = e^{kx}$$

$$\ln(.4) = kx$$

$$x = \frac{\ln(.4)}{k} \approx 38.336 \text{ yrs}$$

$$A) y = 100 e^{\frac{x \ln(0.5)}{29}}$$

$$B) y(14) = 71.56\%$$

$$C) 12.5 = 100 e^{kx}$$

$$\frac{12.5}{100} = e^{kx}$$

$$\frac{1}{8} = e^{kx}$$

$$\ln\left(\frac{1}{8}\right) = kx$$

$$x = \frac{1}{k} \ln\left(\frac{1}{8}\right)$$

$$\approx 87 \text{ yrs.}$$

Newton's Law of Cooling

Newton's law of cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. Let y = the temperature of the object. A = the temperature of the room, then

$$y' = k(y - A)$$

let $u = y - A$

Then $u' = y'$

$$u' = ku$$

$$u = Ce^{kx}$$

$$y - A = Ce^{kx}$$

$$y = A + Ce^{kx}$$

Int. temp of object
ie $x=0$
 $y = A + C$

$$y' = ky$$

$$\hookrightarrow y = y_0 e^{kx}$$

Example: A tater is taken from the oven when it has reached 190 degrees F and is placed in a room with a temperature of 70 degrees F. If the temp of the tater is 160 degrees F after half an hour, what is its temperature after 45min?

$$y = A + C e^{kx}$$

$$\text{Room} = 70^\circ\text{F} = A$$

$$(0, 190^\circ\text{F})$$

$$\left(\frac{1}{2}, 160^\circ\text{F}\right)$$

find temp if $x = \frac{3}{4}$

$$y = 70 + 120 e^{2x \ln\left(\frac{9}{12}\right)}$$

$$y\left(\frac{3}{4}\right) = 70 + 120 e^{2\left(\frac{3}{4}\right) \ln\left(\frac{9}{12}\right)}$$

$$= 147.94^\circ\text{F}$$

$$y = 70 + C e^{kx}$$

$$190 = 70 + C e^0$$

$$120 = C$$

$$y = 70 + 120 e^{kx}$$

$$160 = 70 + 120 e^{\frac{k}{2}}$$

$$90 = 120 e^{k/2}$$

$$\frac{90}{120} = e^{k/2}$$

$$\ln\left(\frac{9}{12}\right) = \frac{1}{2}k$$

$$k = 2 \ln\left(\frac{9}{12}\right)$$