Sections 4.7: Optimization Problems

Example: Find two numbers whose difference is 65 and whose product is a minimum.

$x = 1^{st}\#$

$y = 2^{nd}\#$

$x - y = 65$

$x = 65 + y$

$P = xy$ min.

$P = (65+y)y$

$P = 65y + y^2$

$P' = 65 + 2y$

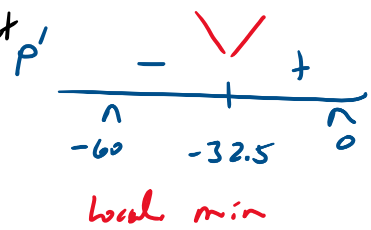
$P' = 0$

$0 = 65 + 2y$

$-65 = 2y$

$y = -32.5$

1st deriv. sign chart



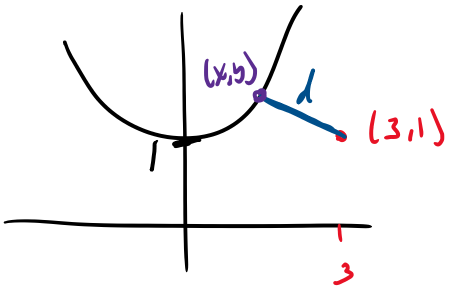
$P'' = 2$ $\quad P''(-32.5) > 0$ i.e. C.U.

Any critical value is a local min.

$y = -32.5$

$x = 65 + y = 32.5$

Example: Find the point on the parabola $y = x^2 + 1$ that is closest to the point $(3, 1)$



min the distance from (x,y) to $(3,1)$
and $y = x^2+1$

$$d = \sqrt{(x-3)^2 + (y-1)^2}$$

$$d = \sqrt{(x-3)^2 + (x^2+1-1)^2} = \sqrt{(x-3)^2 + x^4}$$

Lets minimize the squared distance i.e. $D = d^2$

$$D = (x-3)^2 + x^4 = x^2 - 6x + 9 + x^4 = x^4 + x^2 - 6x + 9$$

$$D' = 4x^3 + 2x - 6$$

$$0 = 4x^3 + 2x - 6$$

$$0 = (x-1)\underbrace{(4x^2+4x+6)}_{\text{Irreducible quadratic.}}$$

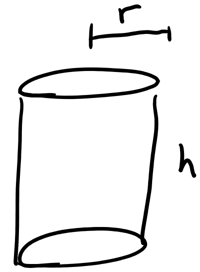
only C.V. is $x=1$

$$D'' = 12x^2 + 2$$

$D''(1) = 14 > 0$ The critical value is a local min.

point $(1, 2)$

Example: A circular cylindrical metal container, open at the top, is to have a capacity of 192π in^3. the cost of the material used for the bottom of the container is 15 cents per in^2, and that of the material used for the side is 5 cents per in^2. If there is no waste of material, find the dimensions that will minimize the cost of the material.



$V = 192\pi$

$V = \pi r^2 h$

$192\pi = \pi r^2 h$

$\frac{192\pi}{\pi r^2} = h$

$h = \frac{192}{r^2}$

$C = 15(\pi r^2) + 5(2\pi r h)$

$C = 15\pi r^2 + 10\pi r h$

$C = 15\pi r^2 + 10\pi r \cdot \frac{192}{r^2}$

$C = 15\pi r^2 + \frac{1920\pi}{r}$

domain $r > 0$

$C = 15\pi r^2 + 1920\pi r^{-1}$

$C' = 30\pi r - 1920\pi r^{-2}$

$C'' = 30\pi + 3840\pi r^{-3}$

$C'' = 30\pi + \frac{3840\pi}{r^3}$ > 0 for $r > 0$

any C.V. is a local min

$C' = 0$

$30\pi r - \frac{1920\pi}{r^2} = 0$

$30\pi r = \frac{1920\pi}{r^2}$

$30\pi r^3 = 1920\pi$

$r^3 = \frac{1920\pi}{30\pi} = \frac{192}{3}$

$r^3 = 64$

$r = 4$

$h = \frac{192}{r^2} = \frac{192}{16}$

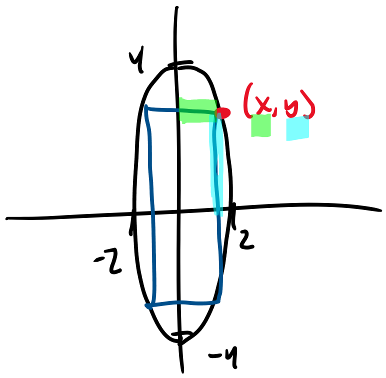
$h = 12$

Example: Find the area of the largest rectangle that can be inscribed in $\frac{x^2}{4} + \frac{y^2}{16} = 1$ $\rightarrow$ $4x^2 + y^2 = 16$

$y^2 = 16 - 4x^2$

$y = \pm\sqrt{16-4x^2}$

$y = \sqrt{16-4x^2}$



$A = (2x)(2y)$

$A = 4xy$

$A = 4x\sqrt{16-4x^2}$

domain is $[0, 2]$

ie $0 \le x \le 2$

$A' = 4\sqrt{16-4x^2} + 4x \cdot \frac{1}{2}(16-4x^2)^{-1/2} \cdot (-8x)$

$A' = 4\sqrt{16-4x^2} - \frac{16x^2}{\sqrt{16-4x^2}}$

$0 = 4\sqrt{16-4x^2} - \frac{16x^2}{\sqrt{16-4x^2}}$

multiply both sides by $\sqrt{16-4x^2}$

$0 = 4(16-4x^2) - 16x^2$

$0 = 64 - 16x^2 - 16x^2$

$32x^2 = 64$

$x^2 = 2$

$x = \pm\sqrt{2}$

CV. $x = \sqrt{2}$

$A(0) = 0$

$A(2) = 0$.

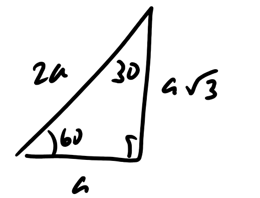
$A(\sqrt{2}) = 4\sqrt{2}\sqrt{16-4(\sqrt{2})^2}$

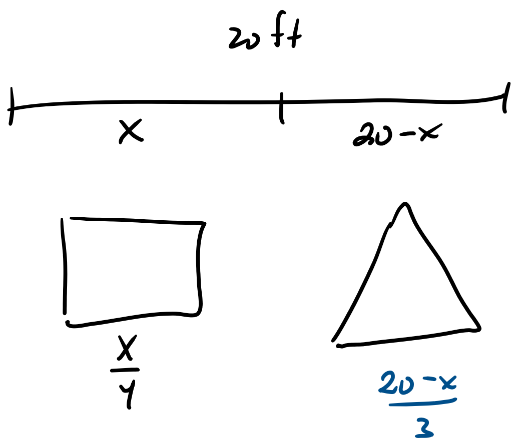
$= 4\sqrt{2}\sqrt{16-4(2)}$

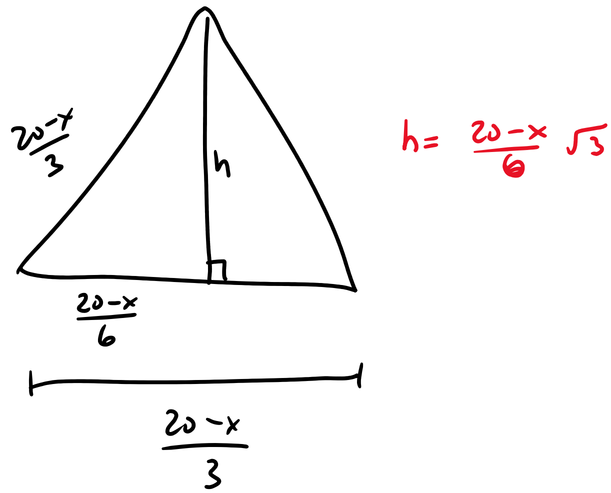
$= 4\sqrt{2}\sqrt{8}$

$= 4\sqrt{16} = 4 \cdot 4 = 16$

Example: A piece of wire 20 feet long is cut into at most two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?







$$A = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{20-x}{3}\right)\left(\frac{20-x}{6}\right)\sqrt{3}$$

$$A = \frac{x^2}{16} + \frac{1}{36}(20-x)^2\sqrt{3}$$

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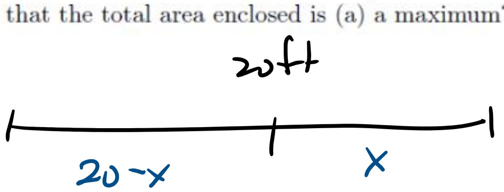
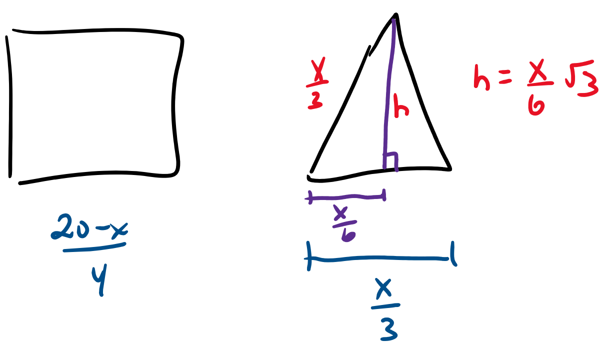




$$A = \left(\frac{20-x}{4}\right)^2 + \frac{1}{2}\cdot\frac{x}{3}\cdot\frac{x}{6}\sqrt{3}$$

$$A = \frac{(20-x)^2}{16} + \frac{x^2}{36}\sqrt{3}$$

Domain is

$0 \le x \le 20$

$$A = \frac{400 - 40x + x^2}{16} + \frac{x^2}{36}\sqrt{3}$$

$$A = \frac{400}{16} - \frac{40x}{16} + \frac{x^2}{16} + \frac{x^2}{36}\sqrt{3}$$

$$A' = -\frac{40}{16} + \frac{2x}{16} + \frac{2x}{36}\sqrt{3}$$

$$A'' = \frac{2}{16} + \frac{2}{36}\sqrt{3} > 0$$

So any C.V. will be a min.

$$0 = -\frac{5}{2} + \frac{x}{8} + \frac{x\sqrt{3}}{18}$$

$$\frac{5}{2} = x\left(\frac{1}{8} + \frac{\sqrt{3}}{18}\right)$$

$$x = \frac{2.5}{\frac{1}{8} + \frac{\sqrt{3}}{18}} = 11.3007 \text{ ft}$$

b) min Area

need 11.3007 ft for the Triangle

need 8.6993 ft for the square.

Test end points

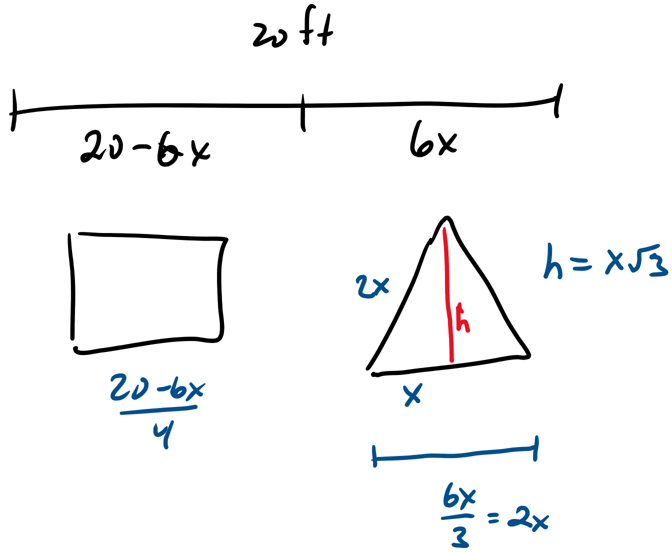
$$A(0) = \frac{20^2}{16} = \frac{400}{16} = 25$$

$$A(20) = \frac{20^2}{36}\sqrt{3} = \frac{400}{36}\sqrt{3} = 19.245$$

Abs max is when all the wire is used for the square.

at most

Example: A piece of wire 20 feet long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) a maximum? (b) a minimum?



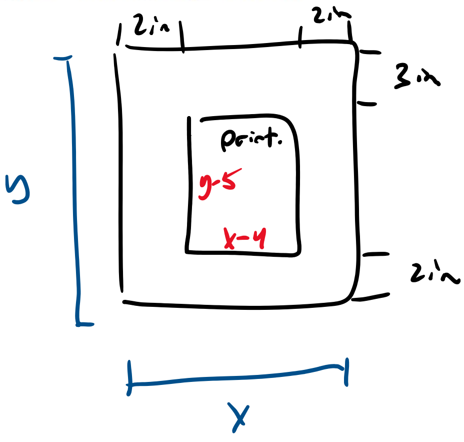
$$A = \left(\frac{20-6x}{4}\right)^2 + \frac{1}{2}(2x)\cdot x\sqrt{3}$$

domain $0 \le x \le \frac{20}{6}$

$$A = \frac{(20-6x)^2}{16} + x^2\sqrt{3}$$

Setup the formula that you would use to solve these problems. Give the domain of the function.

Example: A poster is to have an area of 240 in^2 with a 2-inch margin at the bottom and the sides and a 3-inch margin at the top. What dimensions of the poster will give the largest printed area?



poster

Area = xy

$240 = xy$

$\frac{240}{x} = y$

printed Area

$A = (x-4)(y-5)$

$A = (x-4)\left(\frac{240}{x} - 5\right)$

domain is $x > 4$

$A = 240 - 5x - \frac{960}{x} + 20$

$A = 260 - 5x - \frac{960}{x}$

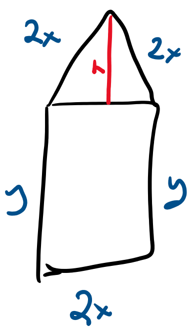
⋮

$x = \sqrt{192}$

$y = \frac{240}{\sqrt{192}}$

Example: A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12ft, find the dimensions of the rectangle that will produce the largest area for the window.

$h = x\sqrt{3}$

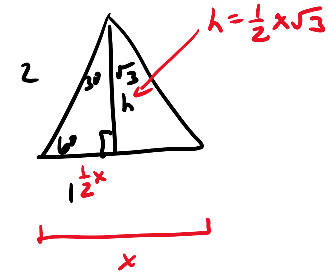


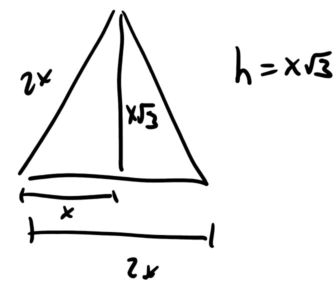
$A = 2xy + \frac{1}{2}(2x)(x\sqrt{3})$

$A = 2xy + x^2\sqrt{3}$

$A = 2x(6-3x) + x^2\sqrt{3}$

$A = 12x - 6x^2 + x^2\sqrt{3}$





$p = 12$

$12 = 2(y) + 3(2x)$

$12 = 2y + 6x$

$6 = y + 3x$

$y = 6 - 3x$

Answer

base $2x = 2.8116$ ft

height $y = 3.672$ ft