Sections 4.9: Antiderivatives

Definition: A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example: Is the function $F = x\ln(x) - x$ an antiderivative of $f = \ln(x)$?

yes

:)

$$F' = 1\ln(x) + x\frac{1}{x} - 1$$

$$= \ln(x) + 1 - 1$$

$$= \ln(x)$$

Example: Find an antiderivative of $f = 2x$.

$x^2 + 6$

$x^2 + 22$

$x^2 + \sqrt{5}$

$x^2 + 42$

$x^2 + C$

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

Table of Antidifferentiation Formulas

Function	Antiderivative
$cf(x)$	$cF(x) + K$
$f(x) \pm g(x)$	$F(x) \pm G(x) + C$
x^n, if $n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\underbrace{x^{-1} = \frac{1}{x}}$	$\ln\lvert x\rvert + C$
e^{kx}	$\frac{1}{K} e^{Kx} + C$
b^{kx}	$\frac{1}{K} b^{Kx} \cdot \frac{1}{\ln(b)} + C$

$y = \ln(x)$ $x>0$

$y' = \frac{1}{x}$

Function	Antiderivative
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\frac{1}{x^2+1}$	$\arctan(x) + C = \tan^{-1}(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$

Example: Find the most general antiderivative.

A) $f(x) = 7x^4 + 3x^2 + 7$

$$F(x) = \frac{7x^5}{5} + \frac{3x^3}{3} + 7x + C$$

$$= \frac{7}{5}x^5 + x^3 + 7x + C$$

B) $f(x) = \sqrt{x} + \sqrt[3]{x^5} + 3^4 \quad = x^{1/2} + x^{5/3} + 3^4$

$3^4 \rightarrow 81$

$$F(x) = \frac{x^{3/2}}{3/2} + \frac{x^{8/3}}{8/3} + 81x + C$$

$$= \frac{2}{3}x^{3/2} + \frac{3}{8}x^{8/3} + 81x + C$$

Example: Find $f(x)$

A) $f'(x) = x^2(x^5 + 2x)$ $= x^7 + 2x^3$

$$f(x) = \frac{x^8}{8} + \frac{2x^4}{4} + C$$

B) $f'(x) = e^{4x} + \sec(x)\tan(x) + 3^x$

$$f(x) = \frac{1}{4}e^{4x} + \sec(x) + 3^x \cdot \frac{1}{\ln(3)} + C$$

$$(5x^3)^{-1} = 5^{-1}x^{-3} = \frac{1}{5}x^{-3}$$

C) $f'(x) = \frac{3}{x^4} + \frac{1}{5x^3} + \frac{4}{x} + \frac{1}{e^{3x}} + \frac{5}{7^{-x}} = 3x^{-4} + \frac{1}{5}x^{-3} + \frac{4}{x} + e^{-3x} + 5\cdot 7^{x}$

$$f(x) = \frac{3x^{-3}}{-3} + \frac{1}{5}\,\frac{x^{-2}}{-2} + 4\ln|x| + \frac{-1}{3}e^{-3x} + 5\cdot 7^{x}\cdot\frac{1}{\ln(7)} + C$$

$$= -x^{-3} - \frac{1}{10}x^{-2} + 4\ln|x| - \frac{1}{3}e^{-3x} + \frac{5\cdot 7^{x}}{\ln(7)} + C$$

D) $f'(x) = \frac{x^3+2x+7}{x^2} = (x^3+2x+7)x^{-2} = x^3x^{-2} + 2xx^{-2} + 7x^{-2}$

$$= x + 2x^{-1} + 7x^{-2}$$

$$f'(x) = \frac{x^3}{x^2} + \frac{2x}{x^2} + \frac{7}{x^2} = x + \frac{2}{x} + 7x^{-2}$$

$$\boxed{f(x) = \frac{x^2}{2} + 2\ln|x| + \frac{7x^{-1}}{-1} + C}$$

$$= \frac{x^2}{2} + 2\ln|x| - \frac{7}{x} + C$$

E) $f'(x) = -2(1+x^2)^{-1}$ $= \dfrac{-2}{1+x^2}$

$f(x) = -2\arctan(x) + C$

Definition: The vector function $R(t) = X(t)\mathbf{i} + Y(t)\mathbf{j}$ is an antiderivative to $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ if $R'(t) = r(t)$.

Example: Find the most general antiderivative of $r(t) = (3t^2 + 2)\mathbf{i} + (\sec^2(t))\mathbf{j}$.

$$= \langle 3t^2+2,\ \sec^2(t) \rangle$$

$$R(t) = \left\langle \frac{3t^3}{3} + 2t + c_1,\ \tan(t) + c_2 \right\rangle$$

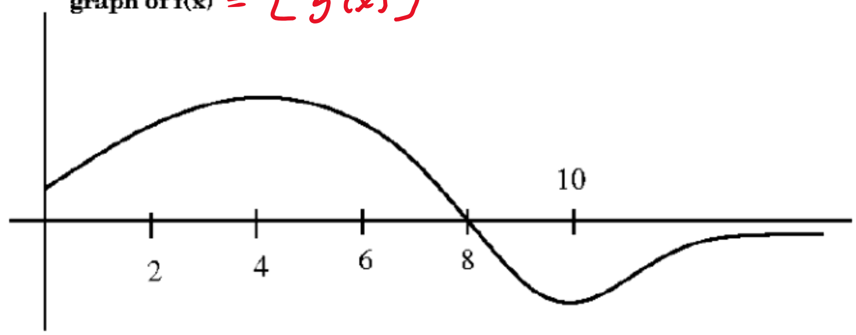
$$= \langle t^3 + 2t + c_1,\ \tan(t) + c_2 \rangle$$

$$R(t) = \langle t^3 + 2t,\ \tan(t) \rangle + C \qquad \text{where } C = \langle c_1, c_2 \rangle$$

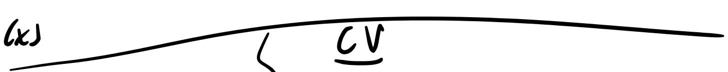
Example: The graph of the function f is given in the figure. Make a rough sketch of the antiderivative F, given that $F(0) = -5$. $\rightarrow$ $g(0) = -5$

sketch $g(x)$

graph of f(x) = $[g'(x)]$



$g(x)$



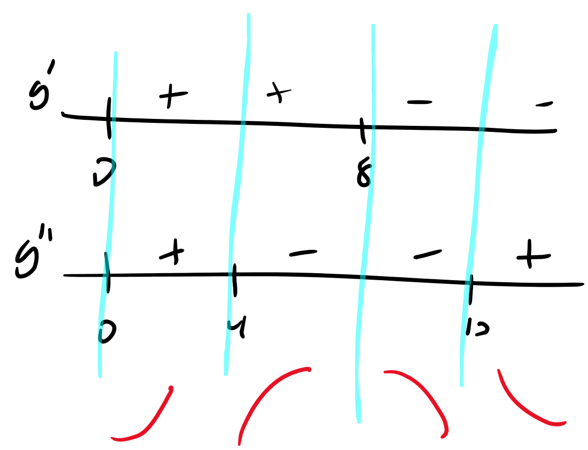
Inc $(0, 8)$
Dec $(8, \infty)$

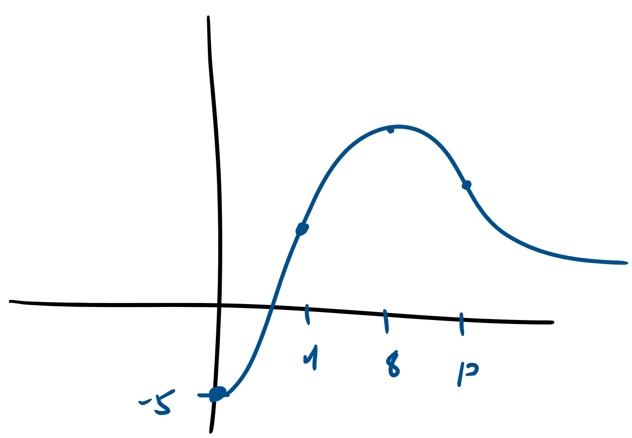
CV
$X = 8$
Local max

$g(x)$

C.U $(0, 4)$ $(10, \infty)$
C.D $(4, 10)$

Inflection pts
$X = 4$
$X = 10$





Example: Find $f(x)$ if $f'(x) = 3x^2 + 15e^{3x} + 4$ given $f(0) = 7$.

$$f(x) = \frac{3x^3}{3} + 15 \cdot \frac{1}{3} e^{3x} + 4x + C$$

$$f(x) = x^3 + 5e^{3x} + 4x + C$$

$$7 = 0 + 5e^0 + 0 + C$$

$$7 = 5 + C$$

$$2 = C$$

$$\boxed{f(x) = x^3 + 5e^{3x} + 4x + 2}$$

Example: Find $f(x)$ if $f''(x) = 20x^3 + 3\sin(x)$ and $f(0) = 2$ and $f'(0) = 8$.

$$f'(x) = \frac{20x^4}{4} - 3\cos(x) + C$$

$$f'(x) = 5x^4 - 3\cos(x) + C$$

$$f(x) = \frac{5x^5}{5} - 3\sin(x) + Cx + d$$

$$f(x) = x^5 - 3\sin(x) + Cx + d$$

$$f(x) = x^5 - 3\sin(x) + 11x + d$$

$$f'(0) = 8$$

$$8 = 0 - 3\cos(0) + C$$

$$8 = -3 + C$$

$$11 = C$$

$$f(0) = 2$$

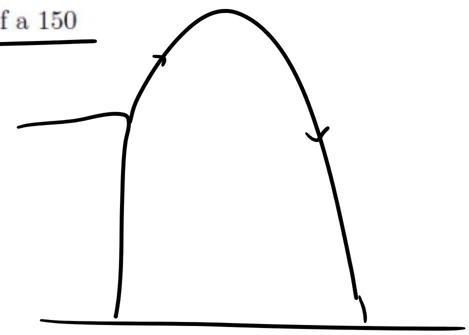
$$2 = 0 - 3\sin(0) + 0 + d$$

$$2 = d$$

$$f(x) = x^5 - 3\sin(x) + 11x + 2$$

Example A ball is thrown upward with a velocity of 50ft/sec from the edge of a 150 foot tall building.

A) Find a formula that gives the height of the ball after x seconds.
B) When does the ball reach its maximum height?
C) How fast does the ball hit the ground?



$v(0) = 50 = C$

$s(0) = 150 = d$

$a(x) = -32 \text{ ft/sec}^2$

$v(x) = -32x + C \longrightarrow v(x) = -32x + 50$

$s(x) = -16x^2 + Cx + d$

$\boxed{s(x) = -16x^2 + 50x + 150}$

b) $v(x) = 0$

$-32x + 50 = 0$

$50 = 32x$

$\boxed{x = \frac{50}{32}} = 1.5625 \text{ sec.}$

c) hits ground at $s(x) = 0$

$0 = -16x^2 + 50x + 150$

quadratic formula to get

$x = -1.875$ ✗

$x = 5$

need speed at $x = 5$

$\text{speed} = |v(5)| = |-32(5) + 50|$

$= |-110 \text{ ft/sec}|$

$= 110 \text{ ft/sec.}$

Example: A car braked with a constant deceleration of $50\text{ft}/\text{sec}^2$, producing skid marks measuring 160ft before coming to a stop. How fast was the car traveling when the brakes were first applied?

↪ find $v(0) = c$

$a(x) = -50 \quad ft/sec^2$

$v(x) = -50x + c$

$s(x) = -25x^2 + cx + d$

$= -25x^2 + cx$

Let $s(0) = 0 \Rightarrow d = 0$

and $s(b) = 160$ ft. so $x = b$ is when the car stops.

$v(b) = 0$

$0 = -50b + c$

$c = 50b$

$160 = s(b)$

$160 = -25b^2 + cb$

$160 = -25b^2 + 50b \cdot b$

$= -25b^2 + 50b^2$

$160 = 25b^2$

$b^2 = \frac{160}{25} = 6.4$

$b = \sqrt{6.4}$

$c = 50\sqrt{6.4}$

$c = 126.49$ ft/sec

Example: A model rocket is launched from the ground. For the first two seconds, the rocket has an acceleration of $a(t) = 12t\ m/sec^2$. At this time all its fuel is spent and it becomes freely falling body.

A) Determine the position function and the velocity function for all times.
B) At what time does the rocket reach its maximum height, and what is that height?

This is a two part problem.

$0 \le t \le 2$ sec. (with fuel)

$a(t) = 12t$

$v(t) = 6t^2 + c_1$

Since $v(0) = 0$ Then $c_1 = 0$

so $v(t) = 6t^2$

$s(t) = 2t^3 + c_2$

Since $s(0) = 0 \Rightarrow c_2 = 0$

So $s(t) = 2t^3$

note $v(2) = 24$ m/s

and $s(2) = 16$ m

These are the position and velocity at the end of the first 2 seconds. They are *also* the "initial" velocity and position of the 2nd part of the flight

$t > 2$ sec. (no fuel)

$a(t) = -9.8$

$v(t) = -9.8t + d_1$

Since $v(2) = 24$

$24 = -9.8(2) + d_1$

$d_1 = 43.6$

$v(t) = -9.8t + 43.6$

$s(t) = -4.9t^2 + 43.6t + d_2$

Since $s(2) = 16$

$16 = -4.9(2)^2 + 43.6(2) + d_2$

$d_2 = -51.6$

$s(t) = -4.9t^2 + 43.6t - 51.6$

Thus.

Thus.

$$\boxed{s(t) = \begin{cases} 2t^3, & 0 \le t \le 2 \\ -4.9t^2 + 43.6t - 51.6, & t > 2 \end{cases}}$$

$$\boxed{v(t) = \begin{cases} 6t^2, & 0 \le t \le 2 \\ -9.8t + 43.6, & t > 2 \end{cases}}$$

max height is when $v(t) = 0$

$6t^2 = 0$

$\Rightarrow t = 0$

doesn't make sense.

$-9.8t + 43.6 = 0$

$t = 4.45$ sec.

max height

$s(4.45) = -4.9(4.45)^2 + 43.6(4.45) - 51.6$

$= 45.39\,m$