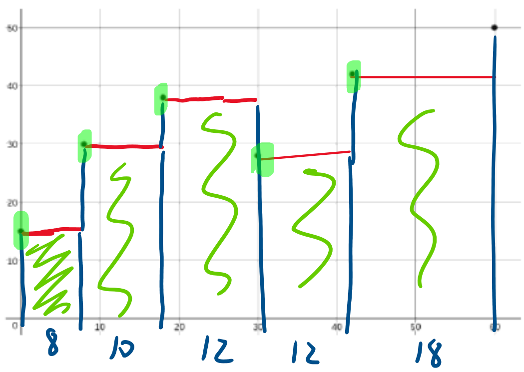
Sections 5.1: Areas and Distances

Example: The velocity of a car is recorded certain times and placed in the following table. Estimate the distance traveled during this first minute.

time (seconds)	0	8	18	30	42	60
velocity (ft/sec)	15	30	38	28	42	50



Left sum

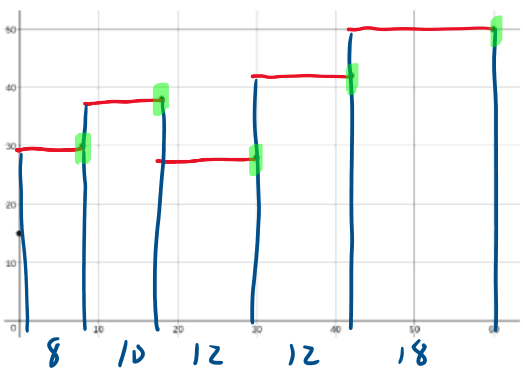
height · base

$15(8-0) + 30(18-8) + 38(30-18) + 28(42-30) + 42(60-42)$

$= 15(8) + 30(10) + 38(12) + 28(12) + 42(18)$

$= 1968$ ft

time (seconds)	0	8	18	30	42	60
velocity (ft/sec)	15	30	38	28	42	50

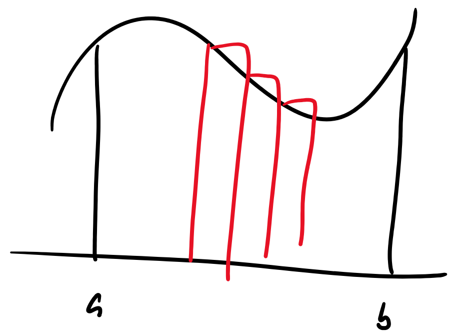


Right sum

base · height

$8(30) + 10(38) + 12(28) + 12(42) + 18(50)$

$= 2360$ ft.



Computing Area under f(x)

Suppose we want to compute the area under $f(x)$ on the interval $[a, b]$ (where $f(x) > 0$ on this inteval).

For a non-linear function, this computation may not be an easy task since the region can not be reduced to geometric figures. We can approximate this area by using a sum of rectangles.

We first need to know the number of rectangles that will be used: n.

Next we need to partition of the given interval into subintervals where each subinterval will become the base of a rectangle. If a partition is given, then use that information. If we have to create the partition, then we choose to have it equally spaced which means that base of each rectangle will be equal.

$$\text{base of each rectangle} = \Delta x = \frac{b-a}{n}$$

The final bit of information required is where to compute the height of the rectangles, sometimes denoted as x_i^*. The three typical styles are left endpoint, right endpoint and midpoint.

General form for x_i^* on the interval $[a, b]$ with equally spaced rectangles.

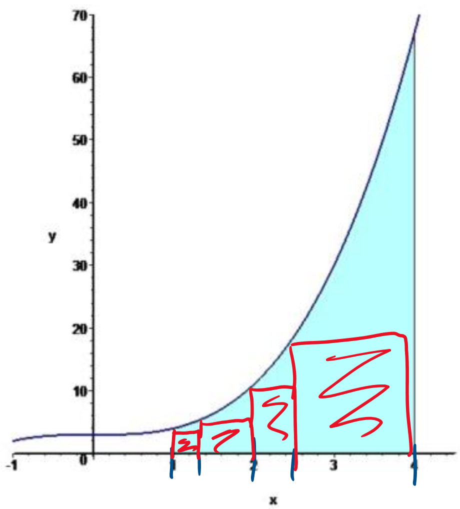
base $= \Delta x = \dfrac{b-a}{n}$ where n is the number of rectangles used.

Right Endpoint:	Left Endpoint:	Midpoint
$x_i^* = a + i\Delta x$	$x_i^* = a + (i-1)\Delta x$	$x_i^* = a + \dfrac{(2i-1)\Delta x}{2}$

Note: When computing the actual area using Riemann sums, we usually use the right endpoint since this method has the easiest formula.

Example: Use the partition $P = \{1,\ 1.2,\ 2,\ 2.5,\ 4\}$ for the interval $[1, 4]$ along with the function $f(x) = x^3 + 2$ to answer the following.

A) Approximate the area under the graph using partition P where x_i^* is the left endpoint of the subintervals. Sketch the rectangles used on the graph.



base $\quad .2 \quad .8 \quad .5 \quad 1.5$

$1 \quad 1.2 \quad 2 \quad 2.5 \quad 4$

$f(x) = x^3 + 2$

base · height

$(.2)\,f(1) + (.8)\,f(1.2) + (.5)\,f(2) + (1.5)\,f(2.5)$

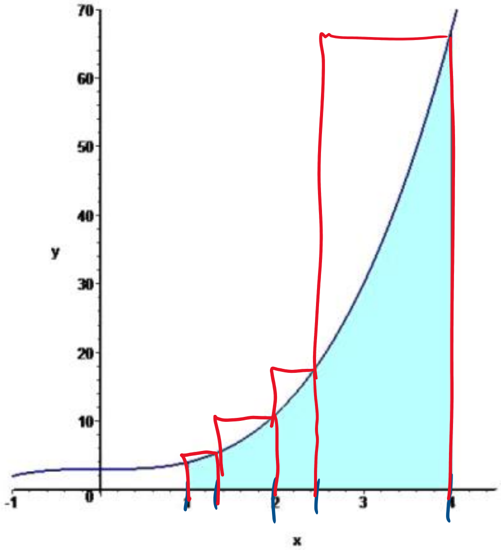
$= (.2)(3) + .8\,(3.728) + .5\,(10) + 1.5\,(17.625)$

$= 35.0199$

underestimate

B) Approximate the area under the graph using the partition $P = \{1,\ 1.2,\ 2,\ 2.5,\ 4\}$ where x_i^* is the right endpoint of the subintervals. Sketch the rectangles used on the graph.

$f(x) = x^3 + 2$



base .2 .8 .5 1.5

1 1.2 2 2.5 4

Base · height

$$\text{Area} = (.2)f(1.2) + (.8)f(2) + (.5)f(2.5) + (1.5)f(4)$$

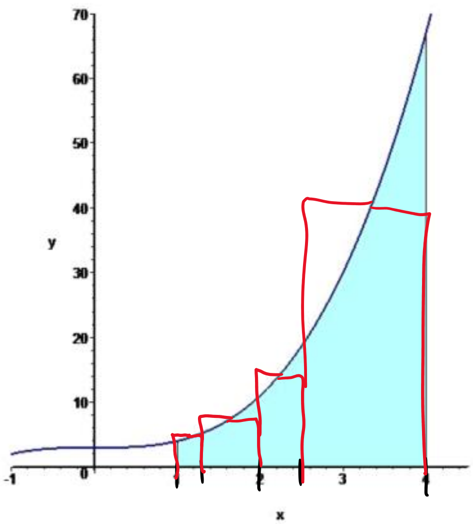
$$= .2(3.728) + (.8)(10) + (.5)(17.625) + 1.5(66)$$

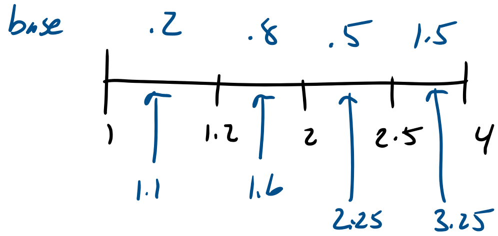
$$= 116.5581$$

over estimate

C) Approximate the area under the graph using the partition $P = \{1,\ 1.2,\ 2,\ 2.5,\ 4\}$ where x_i^* is the midpoint of the subintervals. Sketch the rectangles used on the graph.

$f(x) = x^3 + 2$

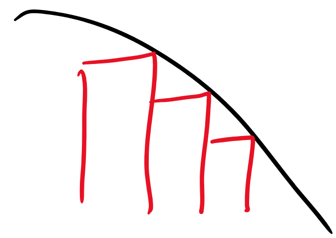




base · height

$$\text{Area} = (.2)\,f(1.1) + (.8)\,f(1.6) + (.5)\,f(2.25) + (1.5)\,f(3.25)$$

$$= 66.73045$$



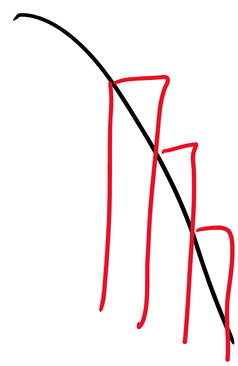
Example: Assume that $f(x)$ is a positive, continuous function on the interval $[a, b]$.

A) What condition would the function $f(x)$ have to have so that the sum of the approximating rectangles will be an underestimate?

Left sum $\Leftrightarrow$ $f(x)$ to be Inc.

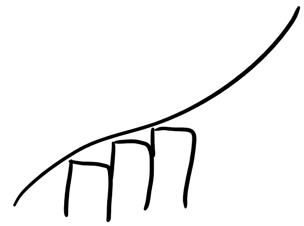
Right sum $\Leftrightarrow$ $f(x)$ is dec

B) What condition would the function $f(x)$ have to have so that the sum of the approximating rectangles will be an overestimate?



Left sum $\Leftrightarrow$ $f(x)$ is dec

Right sum $\Leftrightarrow$ $f(x)$ is inc.



Example: A rocket is launched. The tables gives the velocity(ft/s) of the rocket at time t(sec). Estimate the height of the rocket 32 seconds after liftoff using a left sum. Is your estimate an over estimate or an underestimate?

time (seconds)	0	10	15	20	32	59
velocity (ft/sec)	0	185	320	450	740	1300

$v(t)$ is inc.

left sum

$10(0) + 5(185) + 5(320) + 12(450) = 7925$ ft.

under estimate.

Example: Approximate the area under the function $f(x) = x^2 + 3$ on the interval $[1, 7]$ using a partition that has equal subintervals.

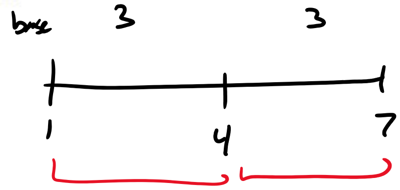
a, b

A) $L_2 =$

left sum

$n = 2$

$\text{base} = \frac{b-a}{n} = \frac{7-1}{2} = \frac{6}{2} = 3$



$\text{Area} = 3f(1) + 3f(4)$

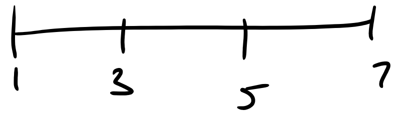
$= 3[f(1) + f(4)] = 3[4 + 15] = 3(23) = 69$

B) $R_3 =$

R = Right

$n = 3$

$\text{base} = \frac{7-1}{3} = \frac{6}{3} = 2$



$\text{Area} = 2f(3) + 2f(5) + 2f(7)$

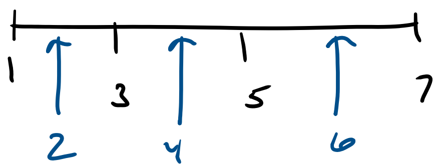
$= 2[f(3) + f(5) + f(7)] = 2[12 + 28 + 52] = 184$

C) $M_3 =$

midpoint

$n = 3$

$\text{base} = \frac{7-1}{3} = \frac{6}{3} = 2$



$\text{Area} = 2f(2) + 2f(4) + 2f(6)$

$= 2[f(2) + f(4) + f(6)] = 2[7 + 19 + 39] = 130$

General form for approximating the area under the curve with rectangles: $\sum_{i=1}^{n} f(x_i^*)\Delta x$

Expression that gives the actual area: $\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$

Example: Express the actual area under the function $f(x) = x^2 + 3$ on the interval $[1, 7]$ using a right sum and partition that has equal subintervals as a Riemann Sum.

a b

$$\text{base} = \Delta x = \frac{b-a}{n} = \frac{7-1}{n} = \frac{6}{n}$$

$$x_i^* = a + i\Delta x = 1 + i\frac{6}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1+\frac{6i}{n}\right)^2 + 3\right]\frac{6}{n}$$

$$\lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left[\left(1+\frac{6i}{n}\right)^2 + 3\right]$$

Example: Set up the Riemann sum that will give the area under the graph for $f(x)$ on the interval $[0, 5]$ using a right endpoint.

$f(x) = x^2 + 7x$

$$\text{base} = \Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

$$x_i^* = a + i\Delta x = 0 + i\frac{5}{n} = \frac{5i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{5i}{n}\right)^2 + 7\left(\frac{5i}{n}\right)\right] \cdot \frac{5}{n}$$

Example: The following represents the area under a function $f(x)$ on an interval $[a,b]$. Find $f(x)$, a, and b. Assume that a right endpoint was used when setting up the summation. Note: There may be more than one right answer.

A) $\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{1+\frac{3i}{n}}$ $= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1+\frac{3i}{n}}$

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

So $b-a=3$

$b=4$

$x_i^* = a + i\Delta x$

$x_i^* = 1 + \frac{3i}{n}$

$a = 1$

Interval $[1,4]$

$f(x) = \sqrt{x}$

$\Delta x = \frac{b-a}{n} = \frac{3}{n}$

$b-a=3$

let $a=0 \to b=3$

$x_i^* = 0 + \frac{3i}{n}$

Interval $[0,3]$

$f(x) = \sqrt{1+x}$

B) $\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} \frac{1}{1+\left(7+\frac{10i}{n}\right)^3}$

$x_i^* = a + i\Delta x$

$= 7 + \frac{10i}{n}$

$a = 7$

$\Delta x = \frac{10}{n} = \frac{b-a}{n}$

$10 = b-a$

$b = 17$

$[7, 17]$

$f(x) = \frac{1}{1+x^3}$