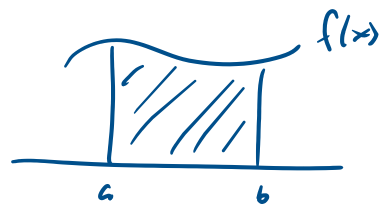
Sections 5.2: The Definite Integral

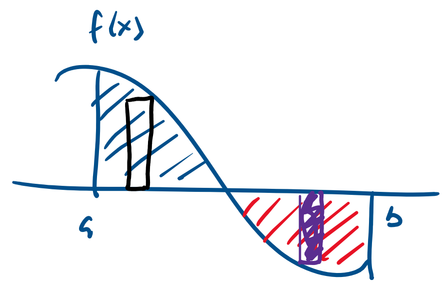
Definition of a Definite Integral: If f is a function on the interval $[a, b]$, we partition the interval $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$. Let x_i^* is any value in the ith subinterval. Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided the limit does exist. If the limit does exist, we say f is **integrable** on the interval $[a, b]$.

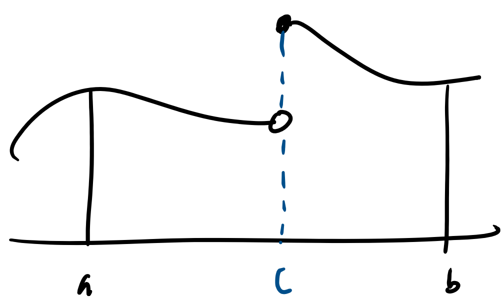


Note 1: If $f(x) \geq 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function f and the x-axis from $x = a$ to $x = b$.



Note 2: If $f(x)$ is not always greater than or equal to zero on the interval $[a, b]$, then the definite integral can be interpreted as the net area on the interval.

Theorem: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.



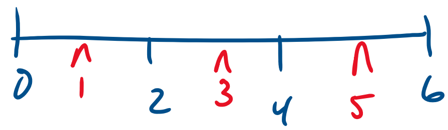
Example: Estimate $\int_0^6 x^2 - 4\, dx$ using a Riemann sum with 3 rectangles with equal bases and the midpoint rule.

↳ $n=3$

$f(x) = x^2-4$

$[a,b] = [0,6]$

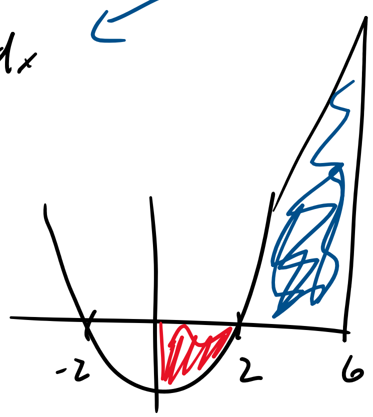
$\text{base} = \frac{6-0}{3} = \frac{6}{3} = 2$



$$\int_0^6 x^2-4\, dx \approx 2\left[f(1) + f(3) + f(5)\right]$$
$$= 2\left[-3 + 5 + 21\right] = 46$$

$\int_0^6 x^2-4\, dx$ is a diff. of Area

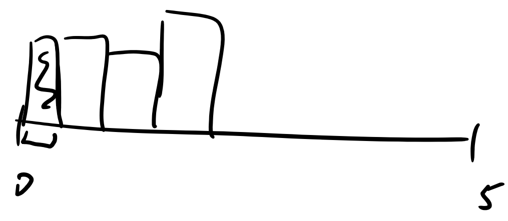
$f(x) = x^2-4$



Area

$$\boxed{\int_2^6 x^2-4\, dx}$$

Example: Suppose that $R(t)$ is the rate, in gallons per hour, that water is pumped into a pool at a water park. Explain the meaning of these integrals.

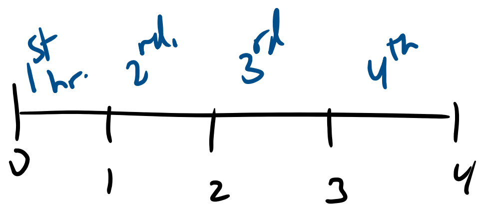


A) $\int_0^5 \underbrace{R(t)}_{\text{rate gallons/hr.}} \, \underbrace{dt}_{\text{time (hrs)}}$

The total # of gallons pumped into the pool in the first 5 hours.

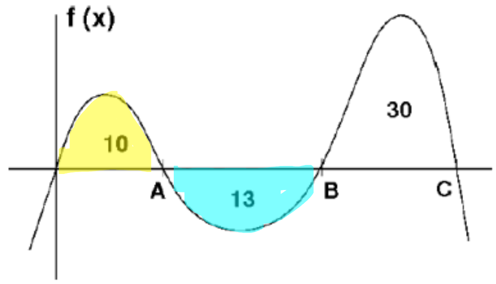
B) $\int_3^4 R(t)\, dt$

The total # of gallons pumped into the pool during the $\underline{4^{th}}$ hr.



Example: Use the graph of f along with the indicated areas to compute these definite integrals.

A) $\int_0^A f(x)\,dx =$ 10

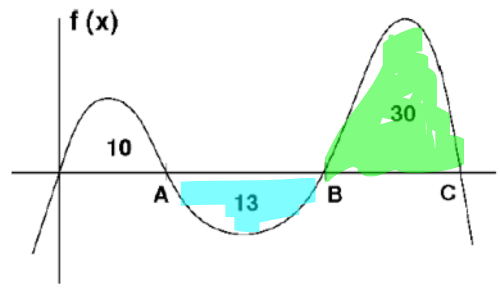


B) $\int_A^B f(x)\,dx =$ -13

C) $\int_0^B f(x)\,dx =$ $10 + -13 = -3$

D) $\int_A^A f(x)\,dx =$ 0

E) $\int_A^C 2f(x)\,dx =$ $2(-13) + 2(30) = -26 + 60 = 34$



F) $\int_A^0 f(x)\,dx =$ -10

G) $\int_B^A f(x)dx =$ $-(-13) = 13$

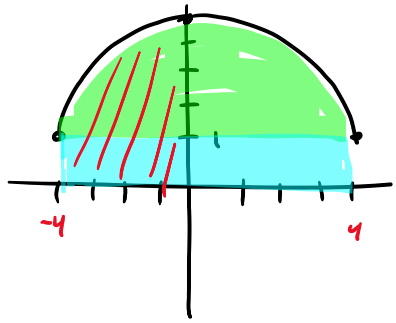
$$\text{base} = \frac{\text{end} - \text{start}}{n} = \frac{0 - A}{n} = \frac{-A}{n}$$

$$\int_A^0 f(x)\,dx = -\int_0^A f(x)\,dx$$

Example: Compute these definite integrals.

A) $\int_{-4}^{4} 1+\sqrt{16-x^2}\;dx$

$f(x) = 1 + \sqrt{16-x^2}$



$\underbrace{y = \sqrt{16-x^2}}$

top of a circle of Radius 4

$y^2 = 16 - x^2$

$x^2 + y^2 = 16$

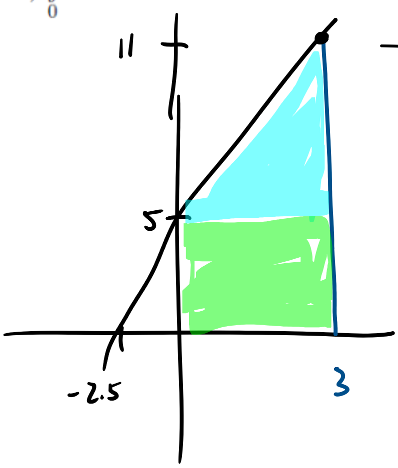
$= \underbrace{\frac{1}{2}\pi(4)^2}_{\text{circle part}} + \underbrace{1(8)}_{\text{Rectangle}}$

$= 8\pi + 8$

A1) $\underbrace{\int_{-4}^{0} 1+\sqrt{16-x^2}\;dx}$ $= \frac{1}{4}(\pi(4)^2) + 1(4)$

$= 4\pi + 4$

B) $\int_{0}^{3} 2x+5\;dx = 15 + \frac{1}{2}(3)(11-5) = 15 + \frac{1}{2}(3)(6) = 15+9 = 24$



$y = 2x+5$

x intercept

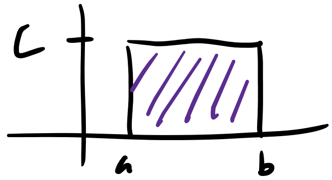
$0 = 2x+5$

$-5 = 2x$

$-2.5 = \frac{-5}{2} = x$

Properties of Definite Integrals

$$\int_a^b c\ dx = c(b-a)$$

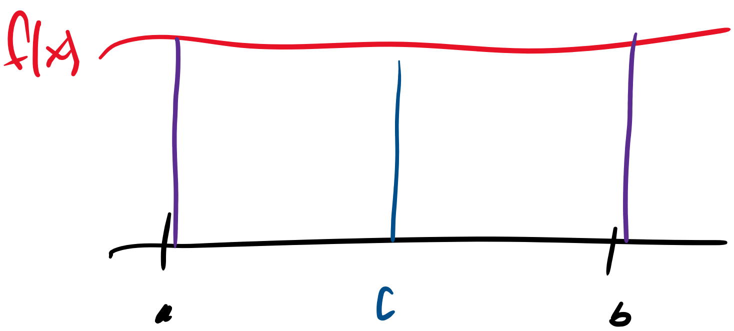


$$\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$$

$$\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$$

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$



Example: If $\int_0^3 f(x)dx = 4$, then evaluate $\int_0^3 (5-2f(x))dx$

$$= \int_0^3 5\,dx - \int_0^3 2f(x)\,dx$$

$$= \int_0^3 5\,dx - 2\int_0^3 f(x)\,dx$$

$$= 5(3-0) - 2(4)$$

$$= 5(3) - 2(4) = 15 - 8 = 7$$

Example: If $\int_1^5 f(x)\,dx = 6$, $\int_1^{10} g(x)\,dx = 10$, and $\int_1^{10} 3f(x) - 4g(x)\,dx = 35$, then compute $\int_5^{10} f(x)\,dx$. $= ?$

$$\int_1^{10} 3f(x)\,dx - \int_1^{10} 4g(x)\,dx = 35$$

$$3\int_1^{10} f(x)\,dx - 4\int_1^{10} g(x)\,dx = 35$$

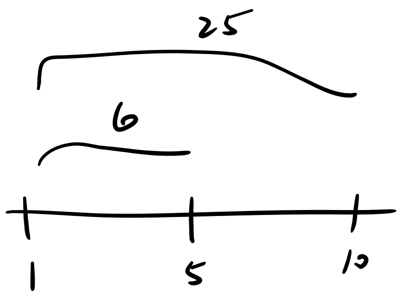
$$3\int_1^{10} f(x)\,dx - 4(10) = 35$$

$$3\int_1^{10} f(x)\,dx - 40 = 35$$

$$3\int_1^{10} f(x)\,dx = 75$$

$$\int_1^{10} f(x)\,dx = 25$$

$$\int_1^{10} f(x)\,dx = 25$$



$$\int_5^{10} f(x)\,dx = \int_1^{10} f(x)\,dx - \int_1^{5} f(x)\,dx$$

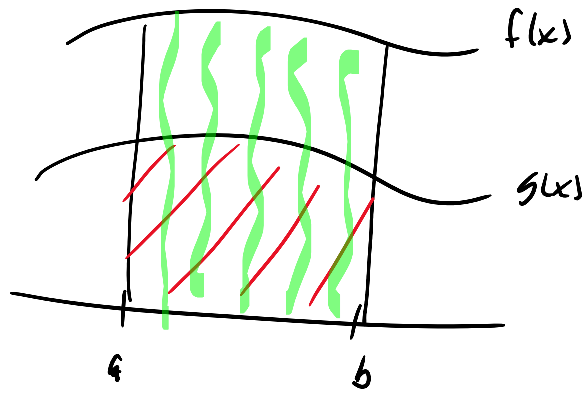
$$= 25 - 6$$

$$= 19$$

Comparison Properties of the Integral

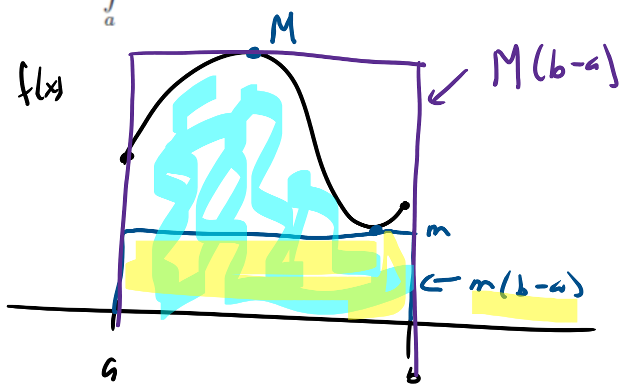
1) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq 0$

2) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$



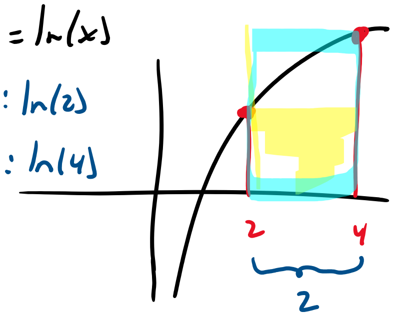
3) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$

Abs min. Abs max

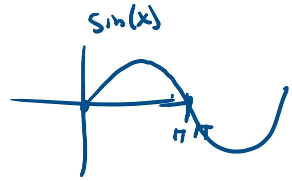


Example: Estimate these definite integrals.

A) $\int_2^4 \ln(x)\,dx$



$$2\ln(2) \le \int_2^4 \ln(x)\,dx \le 2\ln(4)$$



B) $\int_0^{\pi} 2\sin^3(x)+1\,dx$

$f(x) = 2\sin^3(x) + 1$

Abs max = 3

Abs min = 1

$$1(\pi-0) \le \int_0^{\pi} 2\sin^3(x)+1\,dx \le 3(\pi-0)$$