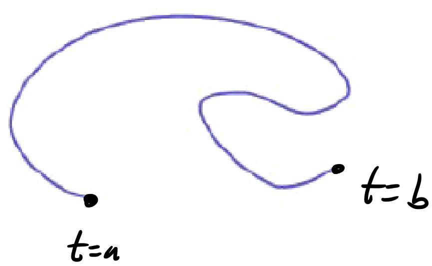
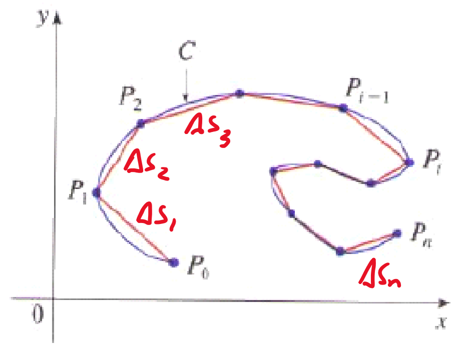
Section 10.2: Calculus with Parametric Functions.

Arc Length

$x = f(t)$
$y = g(t)$

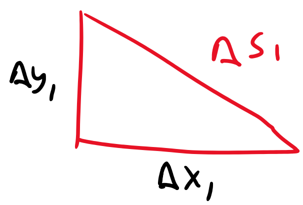
Suppose that C is a smooth curve defined by $x = f(t)$ and $y = g(t)$ for $[a, b]$. Let $\{P_i\}$ be a set of points on the curve that partition of the interval $[a, b]$ such that Δt is equal for each subinterval.





Then the length of the curve(arc length) is given by

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i$$



$$\Delta s_1 = \sqrt{(\Delta x_1)^2 + (\Delta y_1)^2}$$

$$\Delta s_i = |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

$$\Delta s_i = \sqrt{(f'(t_i)\Delta t)^2 + (g'(t_i)\Delta t)^2}$$

$$\Delta s_i = \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\Delta t$$

$f(t) = x$

$f'(t) \approx \dfrac{\Delta x}{\Delta t}$

$\Delta t\, f'(t) \approx \Delta x$

$g(t) = y$

$g'(t) \approx \dfrac{\Delta y}{\Delta t}$

$\Delta t\, g'(t) \approx \Delta y$

$$L = \lim_{n\to\infty} \sum \sqrt{(f'(t_i))^2 + (g'(t_i))^2}\,\Delta t$$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

arclength eq. for 152

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in 251 $\quad x = x(t) \quad y = y(t) \qquad r(t) = \langle x(t), y(t) \rangle$

b $\qquad\qquad$ b

$x = x(t) \quad y = y(t) \qquad r(t) = \langle x(t), y(t) \rangle$

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_a^b |r'(t)|\, dt$$

Example: Find the length of the arc of the curve given by $x(t) = 3t - t^3$, $y(t) = 3t^2$ from the point $(0, 0)$ to the point $(-2, 12)$

$t=0$ $t=2$

$12 = 3t^2$

$4 = t^2$

$\pm 2 = t$

$t=2$ ✓

$x = 6-8 = -2$

$y = 12$

$t=-2$ ✗

$x = -6 - -8 = 2$

$y = 12$

$x' = 3 - 3t^2$

$y' = 6t$

$(3-3t^2)^2 = (3-3t^2)(3-3t^2)$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(3-3t^2)^2 + (6t)^2}$$

$$= \sqrt{9 - 18t^2 + 9t^4 + 36t^2}$$

$$= \sqrt{9 + 18t^2 + 9t^4}$$

$$= \sqrt{(3+3t^2)^2}$$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt = \int_0^2 \sqrt{(3+3t^2)^2}\, dt = \int_0^2 3 + 3t^2\, dt$$

$$= (3t + t^3)\Big|_0^2 = 6 + 8 - (0+0) = 14$$

Example: Find the length of the arc of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$, on the interval $1 \le x \le 2$.

$$\frac{1}{2t} = \frac{1}{2} t^{-1}$$

$$\frac{d}{dt} \frac{1}{2t} = -\frac{1}{2} t^{-2} = \frac{-1}{2t^2}$$

$$x = t \qquad y = \frac{t^3}{6} + \frac{1}{2t}$$

$$1 \le t \le 2$$

$$x' = 1$$

$$y' = \frac{3t^2}{6} - \frac{1}{2t^2} = \frac{t^2}{2} - \frac{1}{2t^2}$$

$$\sqrt{(x')^2 + (y')^2} = \sqrt{(1)^2 + \left(\frac{t^2}{2} - \frac{1}{2t^2}\right)^2}$$

$$= \sqrt{1 + \frac{t^4}{4} - 2 \cdot \frac{t^2}{2} \cdot \frac{1}{2t^2} + \frac{1}{4t^4}}$$

$$= \sqrt{1 + \frac{t^4}{4} - \frac{1}{2} + \frac{1}{4t^4}} = \sqrt{\frac{t^4}{4} + \frac{1}{2} + \frac{1}{4t^4}}$$

$$= \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}$$

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt = \int_1^2 \sqrt{\left(\frac{t^2}{2} + \frac{1}{2t^2}\right)^2}\, dt = \int_1^2 \frac{t^2}{2} + \frac{1}{2t^2}\, dt$$

$$= \left(\frac{t^3}{6} - \frac{1}{2t}\right)\Bigg|_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= \frac{17}{12}$$

Example: Find the length of the arc of the curve $x = 5 - \sqrt{y^3}$, from the point $(4, 1)$ to the point $(-3, 4)$

$1 \le t \le 4$

$x = 5 - \sqrt{t^3}$ $\qquad$ $\underline{\underline{y = t}}$

$x = 5 - t^{3/2}$

$x' = -\frac{3}{2} t^{1/2}$ $\qquad$ $y' = 1$

$$L = \int_1^4 \sqrt{(x')^2 + (y')^2}\, dt = \int_1^4 \sqrt{\left(-\frac{3}{2} t^{1/2}\right)^2 + (1)^2}\, dt$$

$$= \int_1^4 \sqrt{\frac{9}{4} t + 1}\, dt$$

$u = \frac{9}{4} t + 1$

$du = \frac{9}{4}\, dt$ $\qquad$ $\frac{4}{9} du = dt$

$\boxed{t = 1}$ $\quad$ $u = 13/4$

$\boxed{t = 4}$ $\quad$ $u = 10$

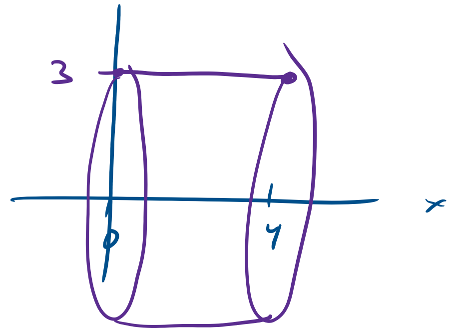
$$= \int_{13/4}^{10} \sqrt{u} \cdot \frac{4}{9}\, du$$

$\sqrt{u} = u^{1/2}$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{13/4}^{10} = \frac{8}{27} (10)^{3/2} - \frac{8}{27} \left(\frac{13}{4}\right)^{3/2}$$

Surface Area

Rotate $y = 3$ from $x = 0$ to $x = 4$ about the x-axis. Find the surface area of the object.

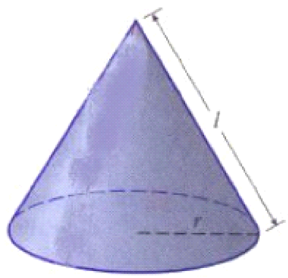


circumference $= \pi d = 2\pi r$

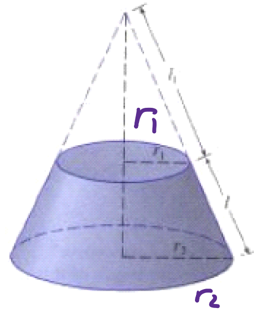
$$\boxed{SA = 2\pi \cdot 3 \cdot 4}$$

$\uparrow$ r $\quad \uparrow$ length

Surface Area of cones.



$SA = \pi r l$

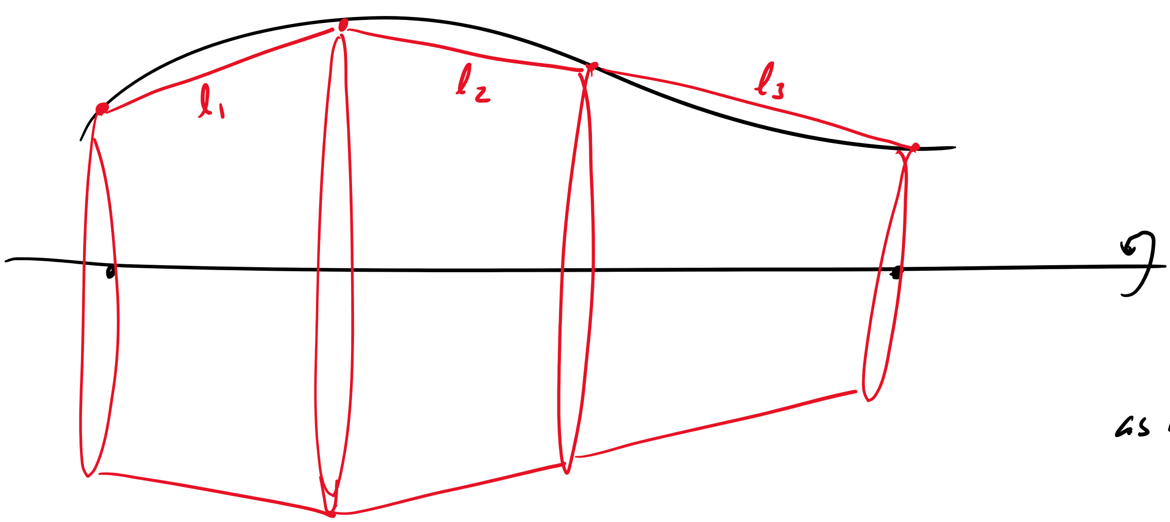


$SA = \pi(r_1 + r_2)l$

let $r = \frac{1}{2}(r_1 + r_2)$

$2r = r_1 + r_2$

$SA = 2\pi r l$



x axis

as $n \to \infty$

$r_* \to r$

$l_i = \Delta s_i$ from before

$$SA = \sum 2\pi r_* l_i$$

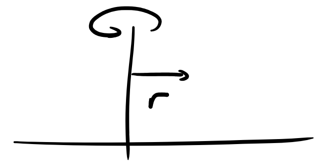
after taking a limit.

$$SA = \int_a^b 2\pi r \sqrt{(x')^2 + (y')^2}\, dt$$

$$SA = \int 2\pi r \sqrt{(x')^2 + (y')^2}\, dt$$

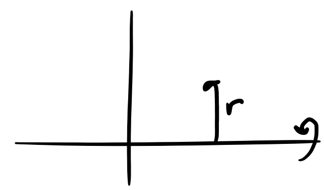
The surface area of a curve rotated about the y-axis:

$$SA = \int_a^b 2\pi x \sqrt{(x')^2 + (y')^2}\, dt$$



The surface area of a curve rotated about the x-axis:

$$SA = \int_a^b 2\pi y \sqrt{(x')^2 + (y')^2}\, dt$$



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Example: Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$ from the point $(1, 1)$ to $(4, 2)$ about the x-axis.

$x = y^2 \quad 1 \le t \le 2$

$x = t^2 \quad y = t$

$1 \le t \le 4$

$x = t \qquad y = \sqrt{t}$

$x' = 1 \qquad y' = \dfrac{1}{2\sqrt{t}}$

$$SA = \int_a^b 2\pi r \sqrt{(x')^2 + (y')^2}\, dt = \int_1^4 2\pi \cdot y \sqrt{(1)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2}\, dt$$

$$= \int_1^4 2\pi \sqrt{t} \sqrt{1 + \frac{1}{4t}}\, dt$$

$$= \int_1^4 2\pi \sqrt{t + \frac{1}{4}}\, dt = 2\pi \, \frac{2}{3} \left(t + \frac{1}{4}\right)^{3/2} \Bigg|_1^4$$

$$= \frac{4\pi}{3} \left[(4.25)^{3/2} - (1.25)^{3/2} \right]$$

Example: Find the area of the surface obtained by rotating the curve $x = t$, $y = \frac{t^2}{4} - \frac{\ln(t)}{2}$ on the interval $1 \le t \le 4$ about the y-axis.

$$SA = \int 2\pi x \sqrt{(x')^2 + (y')^2}\, dt$$

$$x' = 1 \qquad y' = \frac{2t}{4} - \frac{1}{2t} = \frac{t}{2} - \frac{1}{2t}$$

$$SA = \int_1^4 2\pi t \sqrt{1^2 + \left(\frac{t}{2} - \frac{1}{2t}\right)^2}\, dt$$

$$= \int_1^4 2\pi t \sqrt{1 + \frac{t^2}{4} - 2\frac{t}{2}\cdot\frac{1}{2t} + \frac{1}{4t^2}}\, dt$$

$$= \int_1^4 2\pi t \sqrt{1 + \frac{t^2}{4} - \frac{1}{2} + \frac{1}{4t^2}}\, dt$$

$$= \int_1^4 2\pi t \sqrt{\frac{t^2}{4} + \frac{1}{2} + \frac{1}{4t^2}}\, dt = \int_1^4 2\pi t \sqrt{\left(\frac{t}{2} + \frac{1}{2t}\right)^2}\, dt$$

$$= \int_1^4 2\pi t \left(\frac{t}{2} + \frac{1}{2t}\right) dt = 2\pi \int_1^4 \frac{t^2}{2} + \frac{1}{2}\, dt$$

$$= 2\pi \cdot \left(\frac{t^3}{6} + \frac{1}{2}t\right)\Big|_1^4 = 2\pi\left[\frac{4^3}{6} + 2 - \left(\frac{1}{6} + \frac{1}{2}\right)\right]$$

$$= \dots\dots = 24\pi$$