Section 11.10: Taylor and Maclaurin Series

Definition: If a function has a power series representation, then this power series is referred to as the **Taylor series** of the function f at a (or about a or centered at a). If this series is centered at $x = 0$, then this series is given the special name **Maclaurin series**.

Theorem: If $f(x)$ has a power series representation at a, i.e. centered at $x = a$, that is if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with $|x-a| < R$ then its coefficients are given by the formula

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

$$\underbrace{f^{(n)}(a)}_{n^{th} \text{ derivative of } f(x) \text{ at } x=a} = n!\, c_n$$

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$f(x) = \sum_{n=0} c_n (x-a)^n$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

$f(a) = c_0$

$$f'(x) = c_1 + 2*c_2(x-a) + 3*c_3(x-a)^2 + 4*c_4(x-a)^3 + 5*c_5(x-a)^4 + \ldots$$

$f'(a) = c_1$

$$f''(x) = 2*1*c_2 + 3*2*c_3(x-a) + 4*3*c_4(x-a)^2 + 5*4*c_5(x-a)^3 + \ldots$$

$f''(a) = 2 \cdot 1 \cdot c_2$

$f''(a) = 2!\, c_2$

$$f'''(x) = 3*2*1*c_3 + 4*3*2*c_4(x-a) + 5*4*3*c_5(x-a)^2 + 6*5*4*c_6(x-a)^3 \ldots$$

$f'''(a) = 3 \cdot 2 \cdot 1 \cdot c_3$

$f'''(a) = 3!\, c_3$

$$f^{(4)}(x) = 4*3*2*1*c_4 + 5*4*3*2*c_5(x-a) + 6*5*4*3*c_6(x-a)^2 + \ldots$$

$f^{(4)}(a) = 4 \cdot 3 \cdot 2 \cdot 1\, c_4$

$f^{(4)}(a) = 4!\, c_4$

$$f^{(5)}(x) = 5*4*3*2*1*c_5 + 6*5*4*3*2*c_6(x-a) + \ldots$$

$f^{(5)} = 5!\, c_5$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = e^{2x}$

↳ centered at $x=0$ $(a=0)$

need formula for $c_n = \dfrac{f^{(n)}(a)}{n!}$

$f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

$f^{(n)}(x) = 2^n e^{2x}$

$f^{(n)}(0) = 2^n e^0 = 2^n$

$c_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{2^n}{n!}$

$a=0$

$f(x) = \sum\limits_{n=0} c_n (x-a)^n$

$e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{2^n}{n!}(x-0)^n = \sum\limits_{n=0}^{\infty} \dfrac{2^n}{n!} x^n$

Ratio test

$\lim\limits_{n\to\infty} \left| \dfrac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n x^n} \right| = \lim\limits_{n\to\infty} \left| \dfrac{2 \cdot x}{n+1} \right| = 0$

$R = \infty$

$I = (-\infty, \infty)$

$c=0$ (ie $x=0$) centered

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \sin(2x)$

$f(x) = \sin(2x)$ $\quad\quad$ $f(0) = \sin(0) = 0$

$f'(x) = 2\cos(2x)$ $\quad\quad$ $f'(0) = 2\cos(0) = 2$

$f''(x) = -2^2 \sin(2x)$ $\quad\quad$ $f''(0) = 0$

$f'''(x) = -2^3 \cos(2x)$ $\quad\quad$ $f'''(0) = -2^3$

$f^{(4)}(x) = 2^4 \sin(2x)$ $\quad\quad$ $f^{(4)}(0) = 0$

$f^{(5)}(x) = 2^5 \cos(2x)$ $\quad\quad$ $f^{(5)}(0) = 2^5$

$f^{(6)}(x) = -2^6 \sin(2x)$ $\quad\quad$ $f^{(6)}(0) = 0$

$$C_n = \frac{f^{(n)}(0)}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

$f^{(0)}(0) = f(0)$

$$= \frac{f^{(0)}(0)}{0!} + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4 + \cdots$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$$= 0 + \frac{2}{1!}x + \frac{0}{2!}x^2 + \frac{-2^3}{3!}x^3 + \frac{0}{4!}x^4 + \frac{2^5}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-2^7}{7!}x^7 + \cdots$$

$$= 2x + \frac{-2^3}{3!}x^3 + \frac{2^5}{5!}x^5 + \frac{-2^7}{7!}x^7 + \frac{2^9}{9!}x^9 + \frac{-2^{11}}{11!}x^{11} + \cdots$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5$

$$\boxed{\sin(2x) = \sum_{n=0} \frac{(-1)^n \; 2^{2n+1} \; x^{2n+1}}{(2n+1)!}}$$

$2(n+1)+1 = 2n+2+1 = 2n+3$

Radius of conv.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \; 2^{2(n+1)+1} \; x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n \; 2^{2n+1} \; x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} \; 2^{2n+3} \; x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\cancel{(-1)^n} \; 2^{2n+1} \; x^{2n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{2^{2n}} \, 2^3 \, \cancel{x^{2n}} \, x^3}{(2n+3)(2n+2)\cancel{(2n+1)!}} \; \frac{\cancel{(2n+1)!}}{\cancel{2^{2n}} \cdot 2 \; \cancel{x^{2n}} \; x^1} \right|$$

$$\lim_{n\to\infty} \left| \frac{2^2 \cdot x^2}{(2n+3)(2n+2)} \right| = 0 \qquad R = \infty$$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \cos(2x)$

$f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$\cos(2x) = \frac{1}{2} f'(x)$

from last example

$$f(x) = \sin(2x) = \sum_{n=0} \frac{(-1)^n \, 2^{2n+1} \, x^{2n+1}}{(2n+1)!}$$

$$f'(x) = \sum_{n=0} \frac{(-1)^n \, 2^{2n+1} \cdot (2n+1) \, x^{2n}}{(2n+1)!}$$

$$\cos(2x) = \frac{1}{2} \sum_{n=0} \frac{(-1)^n \, 2^{2n+1} \, (2n+1) \, x^{2n}}{(2n+1)!}$$

$(2n+1)! = (2n+1) \cdot (2n)!$

$$\cos(2x) = \sum_{n=0} \frac{(-1)^n \, 2^{2n} \, x^{2n}}{(2n)!}$$

$R = \infty$

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Important Maclaurin series

Note: These are the only building blocks that you do not have to prove the derivation. Any other "building blocks" used, must be proved.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \quad R = \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad R = \infty$$

$$\sin(2x) = \sum_{n=0} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad |x| \le 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad -1 < x \le 1$$

$$\sin(\square) = \sum_{n=0}^{\infty} \frac{(-1)^n \square^{2n+1}}{(2n+1)!}$$

Example: Find the Maclaurin series and the radius of convergence for

A) $f(x) = \sin(3x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!} \qquad R=\infty$$

B) $f(x) = x^2 e^{5x}$

$$= x^2 \sum_{n=0} \frac{(5x)^n}{n!} = \sum_{n=0} \frac{5^n x^{n+2}}{n!} \qquad R=\infty$$

$$\ln(1+x) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1}$$ $|x|<1$ $R=1$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

$\ln(1+(-x))$

$$f(x) = \ln(1+x) - \ln(1-x)$$

$|-x|<1$

$|x|<1$ $R=1$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{(-1)^n (-x)^{n+1}}{n+1}$$

$R=1$

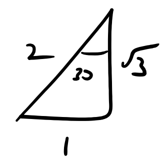
$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{\boxed{(-1)^n (-1)^{n+1}} x^{n+1}}{n+1}$$

$(-1)^n (-1)^{n+1} = -1$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{- x^{n+1}}{n+1}$$

$$= \sum_{n=0} \left(\frac{(-1)^n x^{n+1}}{n+1} + \frac{+ x^{n+1}}{n+1}\right) = \sum_{n=0} \frac{((-1)^n + 1)}{n+1} x^{n+1}$$

Example: Find the Taylor series of $f(x) = \sin(x)$ at $x = \frac{\pi}{6}$



$f(x) = \sin(x)$ $\qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos(x)$ $\qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin(x)$ $\qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos(x)$ $\qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \sin(x)$ $\qquad f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

$$\sin(x) = \frac{\frac{1}{2}}{0!} + \frac{\frac{\sqrt{3}}{2}}{1!}\left(x-\frac{\pi}{6}\right)^1 + \frac{-\frac{1}{2}}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{-\frac{\sqrt{3}}{2}}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{\frac{1}{2}}{4!}\left(x-\frac{\pi}{6}\right)^4 + \frac{\frac{\sqrt{3}}{2}}{5!}\left(x-\frac{\pi}{6}\right)^5 + \cdots$$

Example: Find the Taylor series of $f(x) = \dfrac{1}{x^2}$ about $a = 3$

$$f(x) = \frac{1}{x^2} = x^{-2} = f^{(0)}(x) \Rightarrow n=0 \text{ term}$$

$$f' = -2x^{-3} = \frac{-2}{x^3}$$

$$f'' = 2\cdot 3\, x^{-4} = \frac{2\cdot 3}{x^4}$$

$$f''' = -2\cdot 3\cdot 4\, x^{-5} = \frac{-2\cdot 3\cdot 4}{x^5}$$

$$f^{(4)} = 2\cdot 3\cdot 4\cdot 5\, x^{-6} = \frac{2\cdot 3\cdot 4\cdot 5}{x^6}$$

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$$

does this work for $n \geq 0$? yes

find C_n:

$$C_n = \frac{f^{(n)}(a)}{n!} = \frac{f^{(n)}(3)}{n!} = \frac{1}{n!} \cdot \frac{(-1)^n (n+1)!}{3^{n+2}}$$

$$C_n = \frac{(-1)^n (n+1)\cdot n!}{n! \cdot 3^{n+2}} = \frac{(-1)^n (n+1)}{3^{n+2}}$$

Taylor series $\displaystyle\sum_{n=0}^{\infty} C_n (x-a)^n$

$$\boxed{\frac{1}{x^2} = \sum_{n=0} \frac{(-1)^n (n+1)}{3^{n+2}} (x-3)^n}$$

$0! = 1$

Example: Find the Taylor series of $f(x) = \ln(x)$ about $a = 2$

$f^{(0)}(x) = f(x) = \ln(x)$

$f'(x) = \frac{1}{x} = x^{-1}$

$f'' = -x^{-2}$

$f''' = 2x^{-3}$

$f^{(4)} = -2 \cdot 3x^{-4}$

$f^{(5)} = 2 \cdot 3 \cdot 4x^{-5}$

$f^{(6)} = -2 \cdot 3 \cdot 4 \cdot 5x^{-6} = \frac{-2 \cdot 3 \cdot 4 \cdot 5}{x^6}$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

does it work for all n (i.e. $n \geq 0$)?

no! not valid for $n = 0$

$$\boxed{\frac{(n-1)!}{n!} = \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}}$$

$$C_n = \frac{1}{n!} f^{(n)}(2) = \frac{1}{n!} \frac{(-1)^{n+1}(n-1)!}{2^n}$$

$$C_n = \frac{(-1)^{n+1}}{n 2^n}$$

works for $n \geq 1$

$$f(x) = \ln(x) = \sum_{n=0}^{\infty} C_n (x-2)^n = C_0 + \sum_{n=1}^{\infty} C_n (x-2)^n$$

$$= \underline{\ln(2)} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} (x-2)^n$$

this is $C_0 = \frac{f^{(0)}(2)}{0!} = \frac{f(2)}{0!} = f(2)$

Example: Find the Taylor series of $f(x) = \dfrac{1}{\sqrt{x}}$ about $\underline{a = 4}$

$$f(x) = x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$f' = -\frac{1}{2} x^{-3/2} = \frac{-1}{2 \cdot x^{3/2}}$$

$$f'' = \frac{1}{2} \cdot \frac{3}{2} \cdot x^{-5/2} = \frac{1 \cdot 3}{2^2 x^{5/2}}$$

$$f''' = -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} x^{-7/2} = \frac{-1 \cdot 3 \cdot 5}{2^3 x^{7/2}}$$

$$f^{(4)} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot x^{-9/2} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 x^{9/2}}$$

$$f^{(n)} = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \, x^{\frac{2n+1}{2}}}$$

works for $n \geq 1$.

ie fix for $n = 0$.

$$C_n = \frac{f^{(n)}(4)}{n!} = \frac{1}{n!} \cdot f^{(n)}(4) = \frac{1}{n!} \cdot \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2^n \, 4^{\frac{2n+1}{2}}}$$

$n \neq 0$

$$4^{\frac{2n+1}{2}} = (\sqrt{4})^{2n+1} = 2^{2n+1}$$

$n \neq 0$

$$C_n = \frac{(-1)^n \, 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n! \; 2^{3n+1}}$$

$$\frac{1}{\sqrt{x}} = \underbrace{C_0}_{n=0} + \sum_{n=1}^{\infty} C_n (x-4)^n$$

$$C_0 = \frac{f^{(0)}(4)}{0!} = \frac{f(4)}{1} = \frac{1}{\sqrt{4}}$$

$$= \frac{1}{2} + \sum_{n=1} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n! \; 2^{3n+1}} (x-4)^n$$

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Example: If $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n4^n}$, find $f^{(48)}(3)$.

$$= \sum_{n=0} c_n (x-3)^n$$

$a = 3$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$c_n = \frac{1}{n4^n}$$

$$c_n = \frac{f^{(n)}(3)}{n!}$$

$$\frac{f^{(n)}(3)}{n!} = \frac{1}{n4^n}$$

$$f^{(n)}(3) = \frac{n!}{n4^n}$$

$$f^{(48)}(3) = \frac{48!}{48 \cdot 4^{48}} = \frac{47!}{4^{48}}$$

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Example: Use series to evaluate this integral.

$$\int \frac{e^{x^2} - 1 - x^2}{x} dx$$

$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \frac{x^0}{0!} + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots$$

$$e^{x^2} - 1 - x^2 = (\quad) - 1 - x^2$$

$$= \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{n=2} \frac{x^{2n}}{n!}$$

$$\int \frac{e^{x^2} - 1 - x^2}{x} dx = \int \frac{1}{x} \sum_{n=2} \frac{x^{2n}}{n!} dx = \int \sum_{n=2} \frac{x^{2n-1}}{n!} dx$$

$$C + \sum_{n=2} \frac{x^{2n}}{(2n) \cdot n!}$$

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$$\cos(x) = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example: Find the sum of these series.

(A) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!} = \sum_{n=0} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!}$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

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$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

(B) $\sum_{n=2}^{\infty} \frac{5^n x^{3n}}{n!}$ $= \sum_{n=2} \frac{5^n (x^3)^n}{n!} = \sum_{n=2} \frac{(5x^3)^n}{n!} = e^{5x^3} - \underbrace{\frac{(5x^3)^0}{0!}}_{n=0} - \underbrace{\frac{(5x^3)^1}{1!}}_{n=1}$

missing the $n=0$ & $n=1$ terms

$$e^{5x^3} = \sum_{n=0}^{\infty} \frac{(5x^3)^n}{n!}$$

Answer $e^{5x^3} - 1 - 5x^3$

$$f(x) = \frac{1}{x^4} \quad @ \quad x = 3$$

$$f(x) = x^{-4} = \frac{1}{x^4}$$

$$f' = -4x^{-5} = \frac{-4}{x^5}$$

$$f'' = 4\cdot 5x^{-6} = \frac{4\cdot 5}{x^6}$$

$$f''' = -4\cdot 5\cdot 6\ x^{-7} = \frac{-4\cdot 5\cdot 6}{x^7}\cdot\frac{3\cdot 2}{3\cdot 2}$$

$$f^{(4)} = 4\cdot 5\cdot 6\cdot 7\ x^{-8} = \frac{4\cdot 5\cdot 6\cdot 7}{x^8}\cdot\frac{3\cdot 2}{3\cdot 2}$$

$$f^{(n)} = \frac{(-1)^n (n+3)!}{3!\cdot x^{n+4}}$$

works for $n \geq 0$

$$c_n = \frac{1}{n!}\cdot f^{(n)}(3) = \frac{1}{n!}\,\frac{(-1)^n (n+3)!}{3!\,(3)^{n+4}}$$

$$\frac{1}{x^4} = \sum_{n=0}^{\infty} c_n (x-3)^n$$

$$\frac{1}{x^4} = \sum_{n=0} \frac{(-1)^n (n+3)!}{6\cdot 3^{(n+4)}\, n!}(x-3)^n$$

$$\frac{(n+3)!}{n!} = (n+3)(n+2)(n+1)$$