Section 11.4: The Comparison Test

Note: In this section all series have positive terms.

The Comparison Test (Strict Comparison): Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{6}{5n^3+n^2+1} = K$

$5n^3 < 5n^3 + n^2 + 1$

$\frac{1}{5n^3} > \frac{1}{5n^3+n^2+1}$

$\frac{6}{5n^3} > \frac{6}{5n^3+n^2+1}$

$J = \sum_{n=1}^{\infty} \frac{6}{5n^3} = \frac{6}{5} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3}}$

p-series $p=3$

conv.

by the comparison test since J conv. and $J > K$

K will also conv.

B) $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{7^n+5}$ $= K$

$$7^n + 5 > 7^n$$

$$\frac{1}{7^n+5} < \frac{1}{7^n}$$

$$\frac{3^{2n+1}}{7^n+5} < \frac{3^{2n+1}}{7^n}$$

$$J = \sum_{n=1} \frac{3^{2n+1}}{7^n}$$

$$\frac{3^3}{7} + \frac{3^5}{7^2} + \frac{3^7}{7^3} + \cdots$$

$$r = \frac{3^2}{7} = \frac{9}{7} \qquad |r| > 1$$

J diverges.

Comparison test failed.

Try again on Friday.

C) $\sum_{n=1}^{\infty} \frac{1}{5^n - 2} = K$

$J = \sum_{n=1} \frac{1}{5^n}$

geometric $r = \frac{1}{5}$ since $|r| < 1$

J conv.

$5^n - 2 < 5^n$

$\frac{1}{5^n - 2} > \frac{1}{5^n}$

The comparison test fails

Try on Friday

Converge or diverge

$$\sum_{n=1}^{\infty} \frac{3+\sin(2n)}{n} = J$$

$$-1 \le \sin(2n) \le 1$$

$$2 \le 3+\sin(2n) \le 4$$

$$\frac{2}{n} \le \frac{3+\sin(2n)}{n} \le \frac{4}{n}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{2}{n} \qquad \sum_{n=1}^{\infty} \frac{4}{n}}_{\text{p-series}}$$

$p = 1$

diverge.

Since $\sum_{n=1}^{\infty} \frac{2}{n}$ div. by the comparison test J will also div.

Limit Comparison Test(LCT): Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \geq 0$$

If $L > 0$ then both series converge or both series diverge.

If $L = 0$ and $\sum b_n$ converge, then $\sum a_n$ converge.

If $L = \infty$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

(Note: This test is slightly different that the test given in the book.)

$\sum b_n$ is the Known series

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} \frac{1}{5^n - 2}$

Look like $\sum_{n=1}^{\infty} \frac{1}{5^n}$

Geometric $r = \frac{1}{5}$ $|r| < 1$

Conv.

LCT

$$\lim_{n\to\infty} \frac{\frac{1}{5^n-2}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n-2} \overset{L'H}{=} \lim_{n\to\infty} \frac{5^n \ln(5)}{5^n \ln(5) - 0} = \lim_{n\to\infty} 1 = 1$$

LCT says both series do the same.

Since $\sum \frac{1}{5^n}$ conv. we get $\sum \frac{1}{5^n-2}$ conv.

B) $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n^2+2n}-7}$ $= K$

Look like

$$\sum_{n=1} \frac{5}{\sqrt{n^2}} = \sum \frac{5}{n}$$

p-series $p = 1$ d.v.

LCT

$$\lim_{n\to\infty} \frac{\frac{5}{\sqrt{n^2+2n}-7}}{\frac{5}{n}} = \lim_{n\to\infty} \frac{5}{\sqrt{n^2+2n}-7} \cdot \frac{n}{5}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+2n}-7} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2\left(1+\frac{2}{n}\right)}-7}$$

$$= \lim_{n\to\infty} \frac{n}{\left(n\sqrt{1+\frac{2}{n}}-7\right)} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{2}{n}}-\frac{7}{n}} = \frac{1}{\sqrt{1+0}-0}$$

$$= \frac{1}{\sqrt{1}} = 1 > 0$$

by LCT Both series do the same

Since $\sum \frac{5}{n}$ div we also know K will d.v.

C) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Test for div.

$$\lim_{n\to\infty} \frac{\ln(n)}{n^3} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{3n^2}{1}} = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{3n^2}$$

$$= \lim_{n\to\infty} \frac{1}{3n^3} = 0$$

may or may not conv.

LCT

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series $p=3$ conv.

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \ln(n) = \infty$$

LCT did not work.

$\sum \frac{1}{n}$ p-series $p=1$ div.

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(n)}{n^3}\cdot\frac{n}{1} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n}{1}} = \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{2n}$$

$$= \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

LCT Failed again

$\sum \frac{1}{n^2}$ p-series $p=2$ conv.

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln(n)}{n^3}\cdot\frac{n^2}{1} = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

Since $\sum \frac{1}{n^2}$ conv and Limit is Zero. :)

LCT says $\sum \frac{\ln(n)}{n^3}$ will also converge.

LCT Says $\sum \frac{\ln(n)}{n^3}$ will also converge.

D) $\sum_{n=1}^{\infty} \frac{3n^2+5n}{2^n(n^2+1)}$

LCT

$$\lim_{n\to\infty} \frac{\frac{3n^2+5n}{2^n(n^2+1)}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{3n^2+5n}{2^n(n^2+1)} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{3n^2+5n}{n^2+1} = 3$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometric $r=\frac{1}{2}$ since $|r|<1$ conv.

by LCT Both series will conv. (Both do the same)

$$\sum_{n=10}^{\infty} \frac{(n+1)^4}{\sqrt{n^{10}-5n+2}} = K$$

Looks like

$$\sum \frac{n^4}{\sqrt{n^{10}}} = \sum \frac{n^4}{n^5} = \sum \frac{1}{n}$$

$p=1$ div.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^4}{\sqrt{n^{10}-5n+2}}}{\frac{n^4}{\sqrt{n^{10}}}} = \lim_{n\to\infty} \frac{(n+1)^4}{\sqrt{n^{10}-5n+2}} \cdot \frac{\sqrt{n^{10}}}{n^4}$$

$$\lim_{n\to\infty} \frac{(n+1)^4}{n^4} \cdot \frac{\sqrt{n^{10}}}{\sqrt{n^{10}-5n+2}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \sqrt{\frac{n^{10}}{n^{10}-5n+2}}$$

$$\overset{\text{L'H}}{=} 1^4 \cdot \sqrt{1} = 1$$

By LCT Both series do the same.

Since $\sum \frac{1}{n}$ div. we know K will also div.

E) $\sum_{n=2}^{\infty} \frac{5+cos(n)}{\sqrt{n-1}}$

$4 \leq 5+\cos(n) \leq 6$

$\frac{4}{\sqrt{n-1}} \leq \frac{5+\cos(n)}{\sqrt{n-1}} < \frac{6}{\sqrt{n-1}}$ (expected div.)

$\sum \frac{4}{\sqrt{n-1}}$ LCT with $\sum \frac{1}{\sqrt{n}}$ p-series $p=\frac{1}{2}$ div.

$\lim_{n\to\infty} \frac{\frac{4}{\sqrt{n-1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{4\sqrt{n}}{\sqrt{n-1}} = \lim_{n\to\infty} 4\sqrt{\frac{n}{n-1}} = 4\sqrt{1} = 4$

By LCT Both do the same so $\sum \frac{4}{\sqrt{n-1}}$ will div.

by comparison test we get $\sum \frac{5+\cos(n)}{\sqrt{n-1}}$ div.