Section 11.5: Alternating Series

seq.
$a_n = (-1)^n b_n$

An **alternating series** is a series whose terms are alternately positive and negative. The general term, a_n, is of the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ or $a_n = (-1)^{n-1} b_n$, where b_n is a positive number.

The Alternating Series Test (AST): If the alternating series

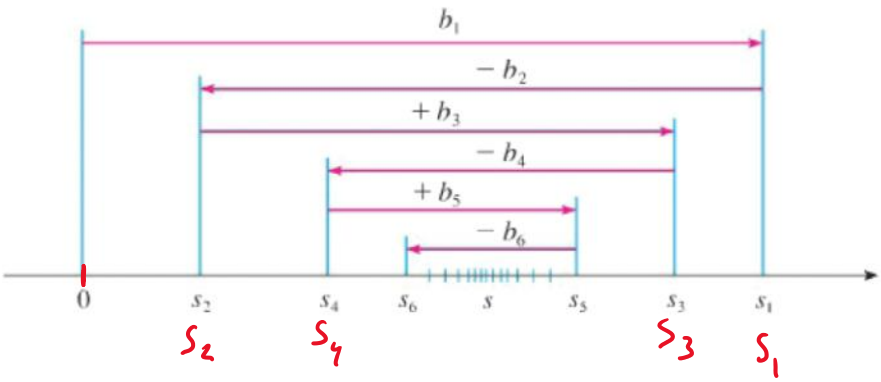
$$\sum_{n=1}^{\infty} (-1)^{(n-1)} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \dots$$

with $b_n > 0$ satisfies:

b_n dec.

(1) $b_{n+1} \le b_n$ for all n and (2) $\lim_{n\to\infty} b_n = 0$ → Test for div.

then the series is convergent.



Note: The Alternating Series Test does not tell us if a series will diverge.

Example: Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$b_n = \frac{1}{n}$

are b_n dec.? yes. ✓

does $\lim_{n \to \infty} \frac{1}{n} = 0$ ✓

By AST This series will converge.

Page 3: estimation thrm.

Alternating Series Estimation Theorem: If $s = \sum_{n=1}^{\infty}(-1)^{(n-1)}b_n$ is the sum of an alternating series that satisfies:

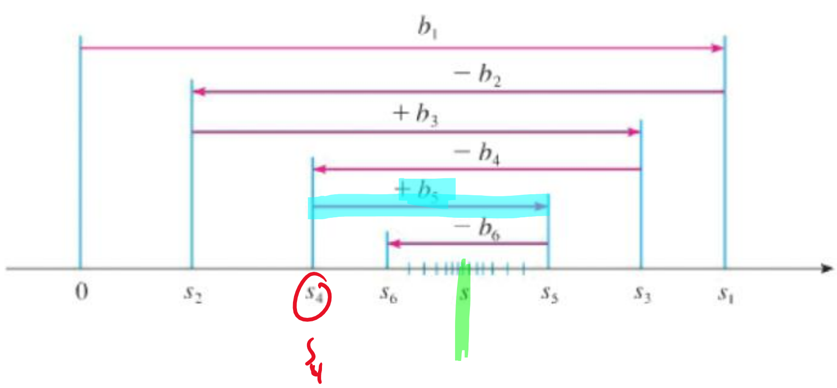
(a) $0 < b_{n+1} \le b_n$ and (b) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

$|R_n| \le$ "next term"

Example: Find a bound on R_4 for the series: $\sum_{n=3}^{\infty} \frac{(-1)^n}{n}$ $= \underbrace{\frac{-1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}}_{S_4} \underbrace{- \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots}_{R_4}$

$|R_4| \le \frac{1}{7}$



Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n^2}\right)$

use AST $\qquad b_n = \ln\left(1+\frac{1}{n^2}\right)$

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right) = \ln(1+0) = \ln(1) = 0 \checkmark$$

Terms dec?

$$f(x) = \ln\left(1+\frac{1}{x^2}\right) = \ln(1+x^{-2})$$

$$f' = \frac{-2x^{-3}}{1+\frac{1}{x^2}} = \frac{\frac{-2}{x^3}}{1+\frac{1}{x^2}} < 0 \quad \text{for } x>0$$

i.e. $f(x)$ dec.

by AST the series converges.

B) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n^2} = \sum_{n=1} \frac{(-1)^{n+1} 3^{n+1}}{n^2}$

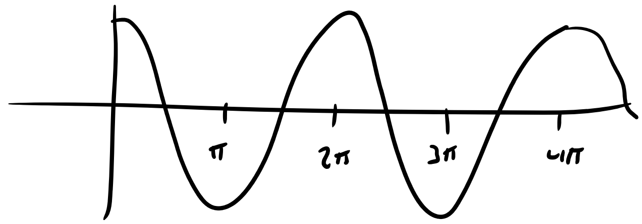
$$b_n = \frac{3^{n+1}}{n^2}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3^{n+1}}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^{n+1} \cdot 1 \cdot \ln(3)}{2n}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{3^{n+1} \cdot 1 (\ln(3))^2}{2} = \infty$$

Thus the series div. by the test for div. !!

C) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{4^n} = \sum \frac{(-1)^n n}{4^n}$



$\cos(n\pi) = (-1)^n$

$b_n = \frac{n}{4^n}$

$\lim_{n\to\infty} \frac{n}{4^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{4^n \ln(4)} = 0$

$f = \frac{x}{4^x} \qquad f' = \frac{4^x \cdot 1 - x \cdot 4^x \ln(4)}{(4^x)^2} = \frac{4^x\left[1 - x\ln(4)\right]}{(4^x)^2}$

$= \frac{1 - x\ln(4)}{4^x} < 0 \quad \text{for } x > \frac{1}{\ln(4)}$

dec. + terms go to zero.

Conv. by AST.

D) $\underbrace{\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2+1}}$

not Alternating.

$\lim_{n \to \infty} \frac{\sin(n)}{n^2+1} = 0$

Test for d.v. says may or may not conv.

wait until the next section.

Assume the series conv. by AST

Example: ~~Determine if the series converges or diverges. If it converges~~ find a bound for the error of s_6, i.e. R_6

$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$$

$b_n = \sqrt{n+1} - \sqrt{n}$

$\boxed{|R_6| \leq b_7}$

$b_7 = \sqrt{8} - \sqrt{7}$

Example: What is the smallest number of terms we must use to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ so that the error is less than $\frac{1}{120}$.

Series conv. by Ast

$b_n = \frac{1}{n^2}$

$|R_n| \le b_{n+1} < \frac{1}{120}$

$b_1 = 1$

$b_2 = \frac{1}{4}$

$b_3 = \frac{1}{9}$

$b_{10} = \frac{1}{100}$

b_{n+1}

$b_{11} = \frac{1}{121}$

$b_{12} = \frac{1}{144}$

smaller Than $\frac{1}{120}$

$n+1 = 11$

$n = 10$

$b_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{120}$

$120 < (n+1)^2 \longrightarrow n+1 = 11$

$\sqrt{120} < n+1$

$n = 10$

$\sqrt{120} - 1 < n$

Conv. by AST

Example: What is the minimum number of terms needed so that the sum of this series is correct to 3 decimal places? i.e. error < 0.0005

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}$$

$$b_n = \frac{1}{(2n-1)!}$$

$b_1 = 1$

$b_2 = .16\bar{6}$

$b_3 = .0083$

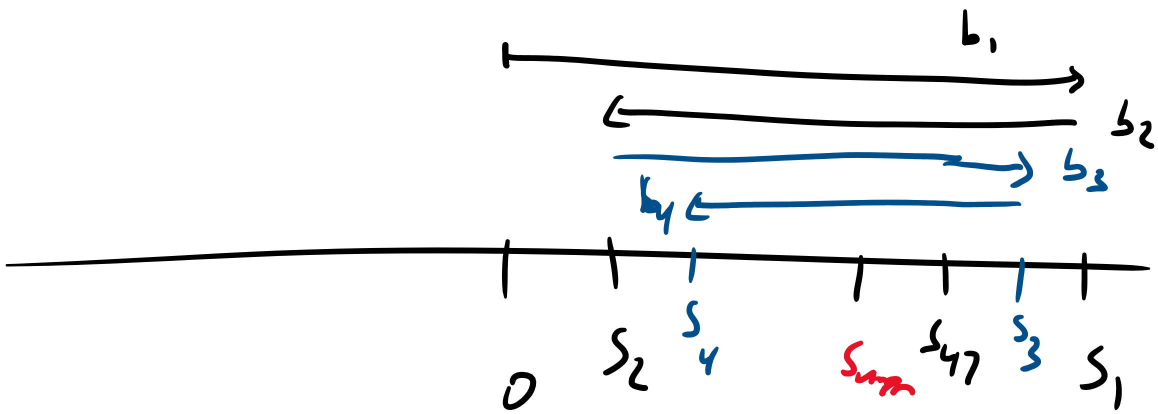
$b_4 = .000198 < .0005 = \text{error}$

next term.

$n = 3$

estimate $= -b_1 + b_2 - b_3$

Example: Given that the alternating series $\sum_{n=1}^{\infty} (-1)^{(n-1)} b_n$ converges. Is the sum of the first 47 terms, s_{47}, an overestimate or an underestimate for the total sum?



over estimate