Section 11.6: Absolute convergence and the ratio and root test

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Definition: A series $\sum a_n$ is called **conditionally convergent** if the series $\sum |a_n|$ is divergent and the series $\sum a_n$ is convergent.

Theorem: If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

Original series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

by The definition of Abs. conv.

The original series converges

New series

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

p-series $p = 3$

conv..

The new series converges

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

Original series

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2+1}$$

new series

$$\sum \left| \frac{\sin(n)}{n^2+1} \right| = \sum \frac{|\sin(n)|}{n^2+1}$$

$$0 \le |\sin(n)| \le 1$$

$$0 \le \frac{|\sin(n)|}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$$

$\sum \frac{1}{n^2}$ p-series $p=2$ conv.

by comparison we get $\sum \frac{|\sin(n)|}{n^2+1}$ conv.

by the def of Abs conv.

the original series will be absolutely conv.

and thus convergent

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

original series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$b_n = \frac{1}{n}$

$b_n \to 0$ as $n \to \infty$

and b_n dec.

by AST. This series conv.

new series

$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ p-series $p=1$ div.

The original series is not Abs. conv.

So the original series is conditionally convergent.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$$

$b_n = \frac{1}{\ln(n)}$

as $n \to \infty$ $b_n \to 0$

b_n dec.

Conv by AST

The original series is conditionally conv.

new series

$$\sum \left| \frac{(-1)^n}{\ln(n)} \right| = \sum \frac{1}{\ln(n)}$$

LCT with $\underline{\sum \frac{1}{n} \text{ div.}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln(n)} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty$$

by LCT $\sum \frac{1}{\ln(n)}$ also div.

The original series is not abs. conv.

The Ratio Test: $\sum a_n$

factorials.

exponentials

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, with $0 \le L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the ratio test is 1, then this test fails to give any information. Try something else.

Consider the results of the ratio test for two of our known $p-$series.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, where $a_n = \frac{1}{n^2}$, converges since $p > 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)^2}{1/n^2}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 1 \text{ by L'Hopitals.}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$, where $a_n = \frac{1}{n}$, diverges since $p \le 1$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)}{1/n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1 \text{ by L'Hopitals.}$$

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

$$a_n = \frac{(-1)^n 3^n}{n!} \qquad a_{n+1} = \frac{(-1)^{n+1} 3^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 3^{n+1}}{(n+1)!}}{\frac{(-1)^n 3^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 3^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n 3^n} \right|$$

$$= \lim_{n\to\infty} \frac{3\, n!}{(n+1)!} = \lim_{n\to\infty} \frac{3\, n!}{(n+1)\, n!}$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = 0$$

$5! = 5\cdot4\cdot3\cdot2\cdot1$

$4! = 4\cdot3\cdot2\cdot1$

$4! = 4\cdot3!$

$(n+1)! = (n+1)\cdot\underbrace{n(n-1)(n-2)\cdots1}$

$= (n+1)\, n!$

by Ratio test the series is Abs. conv.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} n! e^{-n}$$

$$a_n = n!\, e^{-n} \qquad a_{n+1} = (n+1)!\, e^{-(n+1)} = (n+1)!\, e^{-n-1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, e^{-n-1}}{n!\, e^{-n}} \right| = \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!}\ \cancel{e^{-n}}\ e^{-1}}{\cancel{n!}\ \cancel{e^{-n}}}$$

$$= \lim_{n\to\infty} (n+1)\, e^{-1} = \infty$$

The series Diverges by the Ratio test.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(n+1)^3}{(2n)!}$$

$$a_n = \frac{(n+1)^3}{(2n)!} \qquad a_{n+1} = \frac{(n+1+1)^3}{(2(n+1))!} = \frac{(n+2)^3}{(2n+2)!}$$

$$a_{n+1} \cdot \frac{1}{a_n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)^3}{(2n+2)!}\cdot\frac{(2n)!}{(n+1)^3}\right| = \lim_{n\to\infty}\frac{(n+2)^3}{(n+1)^3}\cdot\frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^3\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}$$

$$= \lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^3\cdot\frac{1}{(2n+2)(2n+1)} = 1^3\cdot 0 = 0$$

The series is Abs. conv.

Example: Determine if the series is absolutely convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \frac{(-4)^n n!}{7 * 12 * 17 * ... * (5n+2)}$$

$n=1 \quad 7$

$n=2 \quad 7 \cdot 12$

$n=3 \quad 7 \cdot 12 \cdot 17$

$n=4 \quad 7 \cdot 12 \cdot 17 \cdot 22$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \underbrace{\frac{(-1)^{n+1} \cancel{4^{n+1}}^{\,4} (n+1)!}{7 \cdot 12 \cdot 17 \cdot \ldots \cdot (5n+2) \cdot \underbrace{(5(n+1)+2)}_{5n+7}}}_{a_{n+1}} \cdot \underbrace{\frac{7 \cdot 12 \cdot 17 \cdot \ldots \cdot (5n+2)}{(-1)^n \cancel{4^n}\, n!}}_{\frac{1}{a_n}} \right|$$

$(n+1)!=(n+1)\,n!$

$$= \lim_{n\to\infty} \frac{4 \cdot (n+1)}{5n+7} = \lim_{n\to\infty} \frac{4n+4}{5n+7} = \frac{4}{5} = L$$

Since $0 \le L < 1$ The series is Abs. Conv.

Example: The series $\sum a_n$ is defined recursively by

$$a_1 = 1 \qquad a_{n+1} = \frac{(2+\cos(n))a_n}{\sqrt{n}} \text{ for } n \geq 1.$$

Is the series is absolutely convergent, conditionally convergent, or divergent?

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(2+\cos(n))\,a_n}{\sqrt{n}}}{\frac{a_n}{1}} \right| = \lim_{n\to\infty} \frac{(2+\cos(n))\,a_n}{\sqrt{n}} \cdot \frac{1}{a_n} = 0$$

Series is Abs conv. by Ratio test.

The Root Test: (not covered in this course but is in the textbook)

(a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, with $0 \leq L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** (and therefore convergent).

(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: If the limit for the root test is 1, then this test fails to give any information. Try something else.

Example: Determine if the series is absolute convergent, conditionally convergent, or divergent?

$$\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n+7}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4n+3}{3n+7}\right)^n} = \lim_{n\to\infty} \left(\frac{4n+3}{3n+7}\right) = \frac{4}{3} > 1$$

By the root test the series will diverge.