## Section 11.8: Power Series

---

Definition: A **power series** centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots.$$

where $x$ is a variable and $c_n$ are constants called the coefficients of the series.

---

Example: Where is this power series centered?

$$\sum_{n=0}^{\infty} (2x-10)^n$$

at $x=5$ or $(a=5)$

Example: Is the following a power series?

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$$

centered at $x=0$

$x^n = (x-0)^n$

Geometric series

$a = 1$ $r = x$

Sum $= \frac{a}{1-r} = \frac{1}{1-x}$

converge if $|x| < 1$

$-1 < x < 1$

centered at $x=0$

Interval of convergence

$(-1, 1)$ or $-1 < x < 1$

Radius of convergence

$R = 1$

Pg 2

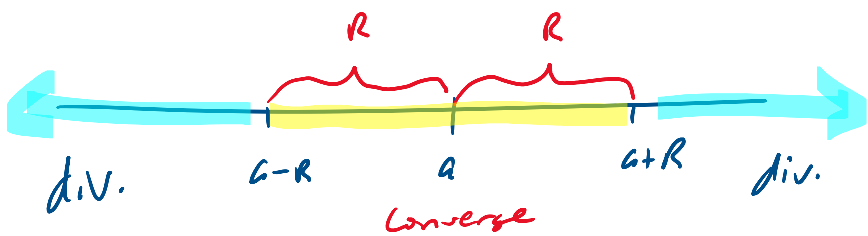
Sunday, October 20, 2019    2:16 PM

**Theorem:** For a given power series, $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities for convergence.

(i) The series converges only when $x = a$    $R = 0$    $I = \{a\}$

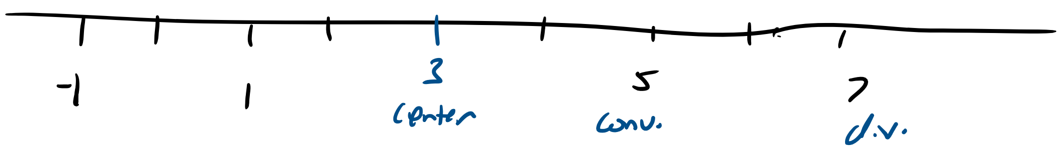
(ii) The series converges for all $x$.    $R = \infty$    $I = (-\infty, \infty)$

(iii) The is a positive number $R$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.




no clue at
$x = a + R$
$x = a - R$
need more work.

Example: Suppose that the series $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges for $x = 5$ and diverges for $x = 7$. For what values of $x$ will this series converge/diverge?




$x=5$ | $R$ is at least 2 $\quad R \geq 2$ $\quad$ we know conv. for $1 < x \leq 5$

$x=7$ | $R$ is at most 4 $\quad R \leq 4$ $\quad$ we know div. for $x \geq 7$

div for $x < -1$

$2 \leq R \leq 4$

no clue about $5 < x < 7$

$-1 \leq x \leq 1$

Sunday, October 20, 2019 2:16 PM

Example: Find the radius and the interval of convergence for the power series.

$$0+\frac{1}{5}x+\frac{2}{5^2}x^2+\frac{3}{5^3}x^3+\frac{4}{5^4}x^4+\ldots = \sum_{n=0}^{\infty}\frac{n}{5^n}x^n$$

Centered at $x=0$

Ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\underbrace{\frac{(n+1)\,x^{n+1}}{5^{n+1}}}_{a_{n+1}}\cdot\underbrace{\frac{5^n}{n\,x^n}}_{\frac{1}{a_n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{5}\cdot\frac{n+1}{n}\right| = \left|\frac{x}{5}\right|$$

conv. if $\left|\frac{x}{5}\right|<1$

$$\frac{|x|}{5}<1$$

$|x-a|<R$

$$|x|<5$$

$|x-0|<5$

centered. R

$$-1<\frac{x}{5}<1$$

Centered at $x=0$

$$-5<x<5$$

$-5<x<5$

$R=5$

$R=5-0=5$

center

$R=\frac{5--5}{2}=5$

$-5<x<5$

To get the Interval of conv. we need to test the end points.

$$\sum_{n=0}\frac{n}{5^n}x^n$$

$\boxed{x=5}$ $\displaystyle\sum_{n=0} \frac{n}{5^n} 5^n = \sum_{n=0} n$ div. by Test for div.

$\lim_{n \to \infty} n = \infty$

$\boxed{x=-5}$ $\displaystyle\sum_{n=0} \frac{n}{5^n} (-5)^n = \sum n(-1)^n$ div. by test for div.

Results $\quad R = 5 \quad I: (-5, 5)$

$3! = 3 \cdot 2 \cdot 1$

$6! = 6 \cdot 5 \cdot 4 \cdot \underbrace{3 \cdot 2 \cdot 1}$
$= 6 \cdot 5 \cdot 4 \cdot 3!$
$= 6 \cdot 5!$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\lim_{n\to\infty} \left| \underbrace{\frac{x^{n+1}}{(n+2)!}}_{a_{n+1}} \cdot \underbrace{\frac{(n+1)!}{x^n}}_{\frac{1}{a_n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^n} \cdot x \quad \cancel{(n+1)!}}{(n+2)\cdot\cancel{(n+1)!} \; \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{n+2} \right| = 0 \leftarrow \text{Series is Abs conv.}$$

does not depend on the value of x.

$R = \infty \qquad I = (-\infty, \infty)$

Pg 6

Sunday, October 20, 2019 2:16 PM

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{\sqrt{n}}$$

centered $x=4$

$$\lim_{n\to\infty} \left| \underbrace{\frac{(x-4)^{n+1}}{\sqrt{n+1}}}_{a_{n+1}} \cdot \underbrace{\frac{\sqrt{n}}{(x-4)^n}}_{\frac{1}{a_n}} \right| = \lim_{n\to\infty} \left| \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{(x-4)^{n+1}}{(x-4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \sqrt{\frac{n}{n+1}} \cdot (x-4) \right| = |x-4|$$

form

$|x-a| < R$

Conv. when $|x-4| < 1$ $\longrightarrow$ $R = 1$

$$-1 < x-4 < 1$$

$$3 < x < 5$$ ← centered at $c=4$ $(x=4)$ $R=1$

Test endpoints $\sum \frac{(x-4)^n}{\sqrt{n}}$

$\boxed{x=5}$ $\sum \frac{(5-4)^n}{\sqrt{n}} = \sum \frac{1^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ p-series $p=\frac{1}{2}$ div.

$\boxed{x=3}$ $\sum \frac{(3-4)^n}{\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$ AST $b_n = \frac{1}{\sqrt{n}}$ dec. $\lim_{n\to\infty} b_n = 0$ conv.

$R = 1$ $\quad I: \; 3 \le x < 5$ or $[3, 5)$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=0}^{\infty} n!(x-1)^n$$

centered at $x=1$ (i.e. $a=1$)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-1)^{n+1}}{n!\,(x-1)^n}\right| = \lim_{n\to\infty}\left|(n+1)\,(x-1)\right| = \begin{cases} 0 & \text{if } x=1 \\ \infty & \text{if } x\neq 1 \end{cases}$$

$R=0 \qquad I=\{1\}$

Example: Find the radius and the interval of convergence for the power series.

$$\sum_{n=0}^{\infty} \frac{n+1}{10^n}(3x-4)^n$$

centered at $x = \frac{4}{3}$

$a_{n+1}$ $\quad \frac{1}{a_n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\,(3x-4)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(n+1)(3x-4)^n}\right|$$

$$\frac{(3x-4)^n(3x-4)10^n}{10^n \cdot 10\,(3x-4)^n}$$

$$= \lim_{n\to\infty}\left|\frac{n+2}{n+1} \cdot \frac{(3x-4)}{10}\right| = \left|1 \cdot \frac{3x-4}{10}\right| = \left|\frac{3x-4}{10}\right|$$

will converge when

$|x-a| < R$

$$\left|\frac{3x-4}{10}\right| < 1$$

$$-1 < \frac{3x-4}{10} < 1$$

$$-10 < 3x-4 < 10$$

$$-6 < 3x < 14$$

$$-2 < x < \frac{14}{3}$$

centered at $a = \frac{4}{3}$

$$R = \frac{14}{3} - \frac{4}{3} = \frac{10}{3}$$

$$R = \frac{4}{3} - -2 =$$

now test end points $\quad \sum \frac{n+1}{10^n}(3x-4)^n$

$x=-2)$ $\quad \sum \frac{n+1}{10^n}(-10)^n = \sum (-1)^n(n+1)$ $\quad$ div. by Test for div.

$x = \frac{14}{3}$ $\quad \sum \frac{n+1}{10^n} \cdot (10)^n = \sum n+1$ div. by test for div.

$R = \frac{10}{3} \qquad I: \left(-2, \frac{14}{3}\right)$