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Section 11.9: Representations of Functions as Power Series

Geometric Power Series: $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$ converges for $|x| < 1$ with Radius of convergence $= 1$ and interval of convergence $(-1, 1)$

$|\square| < 1$

Example: Find the power series representation of $f(x)$ and the radius and interval of convergence.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{n=0}^{\infty} \square^n = \frac{1}{1-\square}$$

conv $\hookrightarrow |\square| < 1$

A) $\dfrac{1}{4+x} = \dfrac{1}{4\left(1+\frac{x}{4}\right)} = \dfrac{1}{4} \cdot \dfrac{1}{1-\boxed{\frac{-x}{4}}}$ → conv. if $\left|\frac{-x}{4}\right| < 1$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4\, 4^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^{n+1}}$$

centered at $\begin{pmatrix} a=0 \\ \text{i.e. } x=0 \end{pmatrix}$

$$\frac{4^{n+1}}{4^{n+2}} = \frac{\cancel{4^n}\cdot 4}{\cancel{4^n}\cdot 4^2} = \frac{4}{4^2} = \frac{1}{4}$$

find Radius & Interval of conv.

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}\, x^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{\cancel{(-1)^n}\, x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{4}\right| = \boxed{\left|\frac{x}{4}\right|} < 1$$

converge when

$a_{n+1} \cdot \frac{1}{a_n}$

$\left|\frac{x}{4}\right| < 1$ → $\frac{|x|}{4} < 1$

$-1 < \frac{x}{4} < 1$

$\underbrace{-4 < x < 4} \hookrightarrow R = 4 - 0 = 4$

$\underbrace{|x| < 4} \rightarrow R = 4$

$-4 < x < 4$

$$\sum_{n=0} \frac{(-1)^n x^n}{4^{n+1}}$$

Test endpoints

$x = 4 \mid$ $\quad \sum \frac{(-1)^n 4^n}{4^{n+1}} = \sum \frac{(-1)^n}{4}$ $\quad b_n = \frac{1}{4}$ $\quad b_n \not\to 0$ so d.v. by test for d.v.

$\boxed{x=4}$ $\quad \sum \frac{(-1)^n 4^n}{4^{n+1}} = \sum \frac{(-1)^n}{4}$ $\quad b_n = \frac{1}{4}$ $\quad b_n \not\to 0$ so div. by test for div.

$\boxed{x=-4}$ $\quad \sum \frac{(-1)^n(-4)^n}{4^{n+1}} = \sum \frac{(-1)^n(-1)^n 4^n}{4^{n+1}} = \sum \frac{1}{4}$ $\quad$ div. by test for div.

$$\frac{1}{4+x} = \sum_{n=0} \frac{(-1)^n x^n}{4^{n+1}} \qquad R = 4 \qquad I: (-4, 4)$$

$$\frac{1}{1-x} = \sum_{n=0} x^n, \quad |x|<1$$

$$\frac{1}{1-\square} = \sum \square^n \quad |\square|<1$$

B) $\frac{x^2}{4+x} = x^2 \cdot \frac{1}{4+x} = x^2 \cdot \frac{1}{4\left(1+\frac{x}{4}\right)}$

$$= \frac{x^2}{4} \cdot \frac{1}{1-\boxed{\frac{-x}{4}}}$$

$$= \underbrace{\frac{x^2}{4} \sum_{n=0}^{\infty} \left(\frac{-x}{4}\right)^n}_{} = \frac{x^2}{4} \sum_{n=0} \frac{(-1)^n x^n}{4^n} = \sum_{n=0} \frac{(-1)^n \cdot x^n \cdot x^2}{4^n \cdot 4}$$

$$\left|\frac{-x}{4}\right| < 1$$

$$\left|\frac{x}{4}\right| < 1$$

$$|x| < 4 \rightarrow R=4$$

$$\boxed{\frac{x^2}{4+x} = \sum_{n=0} \frac{(-1)^n x^{n+2}}{4^{n+1}}}$$

$R=4$

$I: (-4,4)$

$$\frac{1}{1-x} = \sum_{n=0} x^n$$

Example: Find the power series representation of $f(x)$ and the radius of convergence.

A) $\dfrac{3x^3}{1-9x^2} = 3x^3 \cdot \dfrac{1}{1-9x^2} = 3x^3 \cdot \sum_{n=0} (9x^2)^n$

$9^n = 3^{2n}$

$$= 3x^3 \cdot \sum_{n=0} 9^n x^{2n} = \sum_{n=0}^{\infty} 3 \cdot x^3 \cdot 3^{2n} x^{2n}$$

$$= \sum_{n=0}^{\infty} 3^{2n+1} x^{2n+3}$$

$|9x^2| < 1$

$9x^2 < 1$

$x^2 < \frac{1}{9}$

$|x| < \frac{1}{3}$

$R = \frac{1}{3}$ $\quad$ $I: \left(\frac{-1}{3}, \frac{1}{3}\right)$

$$\frac{1}{1-\square} = \sum_{n=0} \square^n \qquad |\square|<1$$

B) $\dfrac{x}{x^2-3x+2}$

$$= \frac{2}{x-2} + \frac{-1}{x-1}$$

$$= \frac{2}{-2+x} - \frac{1}{-1+x}$$

$$= 2 \cdot \frac{1}{-2\left(1-\frac{x}{2}\right)} - \frac{1}{-1\left(1-x\right)}$$

$$= (-1)\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x}$$

$$= \underbrace{(-1)\sum_{n=0}\left(\frac{x}{2}\right)^n}_{} + \underbrace{\sum_{n=0} x^n}_{}$$

converge

$\left|\frac{x}{2}\right|<1$

$|x|<2$

$R=2$

$I \; (-2,2)$

converge.

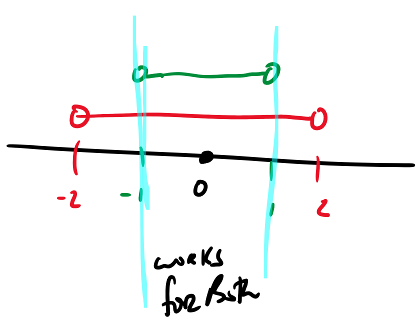
$|x|<1$

$R=1$

$I \; (-1,1)$

$$= \sum_{n=0} \frac{-1x^n}{2^n} + \sum_{n=0} x^n$$

$$= \sum_{n=0}\left(\frac{-x^n}{2^n} + x^n\right) = \sum_{n=0} x^n\left(1-\frac{1}{2^n}\right)$$



$R=1$

$I \; (-1,1)$

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C) $\dfrac{9}{x^4+81} = \dfrac{9}{81+x^4} = \dfrac{9}{81\left(1+\frac{x^4}{81}\right)} = \dfrac{9}{81}\cdot\dfrac{1}{1-\frac{-x^4}{81}}$

$= \dfrac{1}{9}\cdot\dfrac{1}{1-\boxed{\frac{-x^4}{81}}} = \dfrac{1}{9}\sum_{n=0}\left(\dfrac{-x^4}{81}\right)^n$ if $\left|\dfrac{-x^4}{81}\right|<1$

$x^4<81$

$|x|<3$

$R=3$

$= \dfrac{1}{9}\sum_{n=0}\dfrac{(-1)^n x^{4n}}{81^n} = \sum_{n=0}\dfrac{(-1)^n x^{4n}}{9\cdot 81^n}$

$= \sum_{n=0}\dfrac{(-1)^n x^{4n}}{9^{2n+1}}$ since $81^n = 9^{2n}$

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Theorem: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by $\underbrace{f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n}$ is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)dx = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1}$$

The radii of convergence for both $f'(x)$ and $\int f(x)dx$ are both R. The interval of convergence may change.

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$$f = \sum_{n=0}^{\infty} \frac{x^n}{3^n}$$

$$\mathrm{f} = \frac{x^0}{3^0} + \frac{x^1}{3^1} + \frac{x^2}{3^2} + \cdots$$

$$\mathrm{f} = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots$$

$$f' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n}$$

$$g = \sum_{n=0}^{\infty} \frac{x^{n+2}}{3^n}$$

$$\mathrm{g} = \frac{x^2}{3^0} + \frac{x^3}{3^1} + \frac{x^4}{3^2} + \cdots$$

$$\mathrm{g} = x^2 + \frac{x^3}{3^1} + \frac{x^4}{3^2} + \cdots$$

$$g' = \sum_{n=0}^{\infty} \frac{(n+2)x^{n+1}}{3^n}$$

Example: Evaluate this integral by using a power series and find the radius of convergence.

$$\int \frac{9}{x^4+81}\,dx =$$

$$\underbrace{\frac{9}{x^4+81} = \sum_{n=0} \frac{(-1)^n x^{4n}}{9^{2n+1}} \quad R=3 \quad I:(-3,3)}_{}$$

from the previous example.

$$= \int \sum_{n=0} \frac{(-1)^n x^{4n}}{9^{2n+1}}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)\,9^{2n+1}} \qquad \underline{R=3}$$

Test endpoints

$x=3$

$$\sum_{n=0} \frac{(-1)^n 3^{4n+1}}{(4n+1)\cdot 9^{2n+1}} = \sum_{n=0} \frac{(-1)^n 3^{4n+1}}{(4n+1)\,(3^2)^{2n+1}}$$

$$= \sum_{n=0} \frac{(-1)^n 3^{4n+1}}{(4n+1)\,3^{4n+2}} = \sum_{n=0} \frac{(-1)^n}{(4n+1)\cdot 3}$$

$b_n = \frac{1}{12n+3}$

Conv by AST

$(x=-3)$

$$\sum_{n=0} \frac{(-1)^n (-3)^{4n+1}}{(4n+1)\,3^{4n+2}} = \sum_{n=0} \frac{(-1)^n (-1)^{4n+1}\,3^{4n+1}}{(4n+1)\,3^{4n+2}} = \sum \frac{(-1)^{5n+1}}{(4n+1)\cdot 3}$$

$b_n = \frac{1}{12n+3}$

Conv. by AST

$R=3, \quad I=[-3,3]$

Example: Find a power series representation of $f(x)$ and determine the interval and radius of convergence.

$f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x} = \underbrace{\frac{1}{1-(-x)}}_{|-x|<1} = \sum_{n=0} (-x)^n = \sum_{n=0} (-1)^n x^n$$

$R = 1$

$I = (-1, 1)$

$|-x| < 1$

$|x| < 1 \quad R = 1$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0} (-1)^n x^n \, dx = C + \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1}$$

$$\ln(1+x) = C + \boxed{\sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1}}$$

$R = 1$

$C + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Solve for C.

pick $x=0$

$\ln(1+0) = C + 0$

$\ln(1) = C$

$C = 0$

$\int x^3\,dx = \frac{x^4}{4} + C$

$f'(x) = x^3$ and $f(1) = 10$

$f(x) = \frac{x^4}{4} + C$

$10 = \frac{1}{4} + C$

$C = 9.75$

$$\boxed{\ln(1+x) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} \qquad R = 1 \qquad I: (-1, 1]}$$

Building block #2

Example: Find the power series representation of these functions. determine the radius of convergence.

$$\ln(1+x) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} \qquad R=1$$

A) $f(x) = \ln(1-x) = \ln\left(1 + \boxed{(-x)}\right)$

$|-x| < 1$

$|x| < 1$ $\quad R=1$

$$= \sum_{n=0} \frac{(-1)^n (-x)^{n+1}}{n+1} = \sum_{n=0} \frac{(-1)^n (-1)^{n+1} x^{n+1}}{n+1} = \sum_{n=0} \frac{-x^{n+1}}{n+1}$$

$R=1$

B) $f(x) = \ln(4+x^2)$

$$f'(x) = \frac{2x}{4+x^2} = \frac{2x}{4\left(1+\frac{x^2}{4}\right)} = \frac{2x}{4} \cdot \frac{1}{1-\left(\frac{-x^2}{4}\right)} = \frac{x}{2} \sum_{n=0} \left(\frac{-x^2}{4}\right)^n$$

$$= \frac{x}{2} \sum_{n=0} \frac{(-1)^n x^{2n}}{4^n} = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2 \cdot 4^n}$$

$\left|\frac{-x^2}{4}\right| < 1$

$|x^2| < 4$

$|x| < 2$

$R = 2$

$$\ln(4+x^2) = \int f'(x)\,dx = C + \sum_{n=0} \frac{(-1)^n x^{2n+2}}{2 \cdot 4^n \cdot (2n+2)}$$

let $x=0$ $\qquad \ln(4) = C + 0$

$C = \ln(4)$

$$\ln(4+x^2) = \ln(4) + \sum_{n=0} \frac{(-1)^n x^{2n+2}}{2 \cdot 4^n (2n+2)} \qquad R = 2$$

$= 4^{n+1}(n+1)$

$= 4 \cdot 4^n (n+1)$

$2 \cdot 4^n \cdot (2n+2) = 2 \cdot 4^n \cdot 2(n+1)$

$$\ln(4+x^2) = \ln\left[4\left(1+\frac{x^2}{4}\right)\right] = \ln(4) + \ln\left(1+\boxed{\frac{x^2}{4}}\right)$$

$$= \ln(4) + \sum_{n=0} \frac{(-1)^n}{n+1} \cdot \left(\frac{x^2}{4}\right)^{n+1} = \ln(4) + \sum_{n=0} \frac{(-1)^n \, x^{2n+2}}{(n+1)\, 4^{n+1}}$$

$$\left|\frac{x^2}{4}\right| < 1 \longrightarrow |x^2| < 4 \longrightarrow |x| < 2 \qquad R = 2$$

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Example: Find the power series representation of $f(x)$ and determine the radius of convergence.

$f(x) = \arctan(x)$

$$f' = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0} (-x^2)^n = \sum_{n=0} (-1)^n x^{2n}$$

$|-x^2| < 1$

$|x^2| < 1$

$|x| < \sqrt{1} = 1$

$R = 1$

$$\arctan(x) = \int f'(x)\,dx = C + \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$R = 1$

Solve for C.

let $x=0$

$\arctan(0) = C + 0$

$0 = C$

$$\boxed{\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1}$$

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$$f(x) = x^2 \arctan(3x^4) = x^2 \cdot \sum_{n=0} \frac{(-1)^n (3x^4)^{2n+1}}{2n+1}$$

$$|3x^4| < 1$$

$$|x^4| < \frac{1}{3}$$

$$|x| < \sqrt[4]{\frac{1}{3}}$$

$$R = \sqrt[4]{\frac{1}{3}}$$

$$= x^2 \sum_{n=0} \frac{(-1)^n 3^{2n+1} (x^4)^{2n+1}}{2n+1}$$

$$= \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{8n+4} \cdot x^2}{2n+1}$$

$$x^2 \arctan(3x^4) = \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{8n+6}}{2n+1}$$

Example: Find a power series representation of $f(x)$.

$$f(x) = \frac{1}{(1+x)^3}$$

$$g = \frac{1}{1+x} = (1+x)^{-1} = \frac{1}{1-(-x)} = \sum_{n=0} (-x)^n = \sum_{n=0} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$|-x| < 1$

$R = 1$

$f(x) = \frac{1}{2} g''(x)$

$$g' = -(1+x)^{-2} = \frac{-1}{(1+x)^2} = \sum_{n=1} (-1)^n n x^{n-1}$$

$$g'' = 2(1+x)^{-3} = \frac{2}{(1+x)^3} = \sum_{n=2} (-1)^n n(n-1) x^{n-2}$$

$$f = \frac{1}{2} g''(x) = \frac{1}{2} \sum_{n=2} (-1)^n n(n-1) x^{n-2} = \boxed{\sum_{n=2} \frac{1}{2} (-1)^n n(n-1) x^{n-2}}$$

$R = 1$

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$$\frac{1}{(1+x)^3} = \sum_{n=0} \boxed{}$$

$$\sum_{n=0} \frac{1}{2} (-1)^{n+2} (n+2)(n+1) x^n$$

to shift the index

Let $j = n-2$

$j+2 = n$ $\qquad$ $j+1 = n-1$

$$\leftarrow \sum_{j=0} \frac{1}{2} (-1)^{j+2} (j+2)(j+1) x^j$$

Example: Find a power series representation of $f(x)$.

$$f(x) = \frac{x^3}{(1+2x)^3}$$

$$f(x) = \frac{x^3}{8} g''$$

$$g = \frac{1}{1+2x} = (1+2x)^{-1} = \frac{1}{1-(-2x)} = \sum_{n=0} (-2x)^n = \sum_{n=0} (-1)^n 2^n x^n$$

$$= (-1)^0 2^0 x^0 + (-1)^1 2^1 x^1 + \dots$$

$$g' = -(1+2x)^{-2} \cdot 2$$

$$= -2(1+2x)^{-2} = \frac{-2}{(1+2x)^2} = \sum_{n=1} (-1)^n 2^n n x^{n-1}$$

$$g'' = 4(1+2x)^{-3}(2)$$

$$g'' = \frac{8}{(1+2x)^3} = \sum_{n=2} (-1)^n 2^n n(n-1) x^{n-2}$$

$$f = \frac{x^3}{8} g'' = \sum_{n=2} \frac{x^3}{8} \cdot (-1)^n 2^n n(n-1) x^{n-2}$$

$$= \sum_{n=2} (-1)^n 2^{n-3} n(n-1) x^{n+1}$$

$$\frac{2^n}{8} = \frac{2^n}{2^3} = 2^{n-3}$$

Example: Use a series to evaluate this integral.

$$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$R=1$

$|x|<1$

$$\int \arctan(x^3)\, dx = \int \sum_{n=0} \frac{(-1)^n (x^3)^{2n+1}}{2n+1}\, dx$$

$$= \int \sum_{n=0} \frac{(-1)^n x^{6n+3}}{2n+1}\, dx$$

$$= C + \sum_{n=0} \frac{(-1)^n x^{6n+4}}{(2n+1)(6n+4)}$$

$|x^3| < 1$

$|x| < 1$

$R = 1$

Building blocks(so far)

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$$\frac{1}{1-x} = \sum_{n=0} x^n \quad \text{if } |x|<1 \qquad R=1 \qquad I: (-1,1)$$

$$\ln(1+x) = \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} \quad \text{if } |x|<1$$

Test endpoints to find Interval of conv.

$$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1} \quad \text{if } |x|<1$$