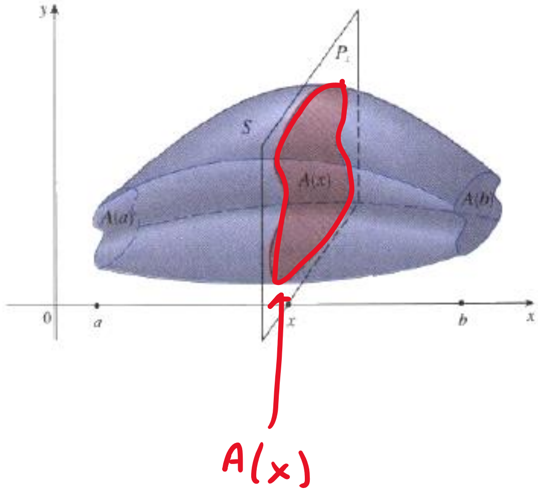
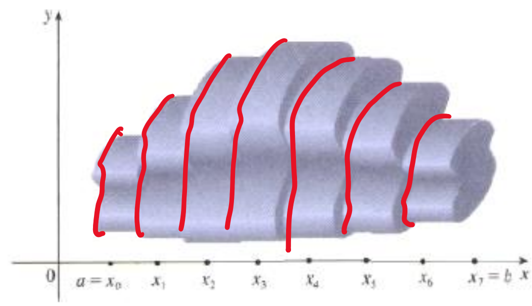
Section 6.2: Volume

Let S be a sold that lies between the planes P_a and P_b. Assume that cross sections of the solid is given by A and are perpendicular to the x-axis.





$$\sum_{i=1}^{n} A(x)\,\Delta x$$

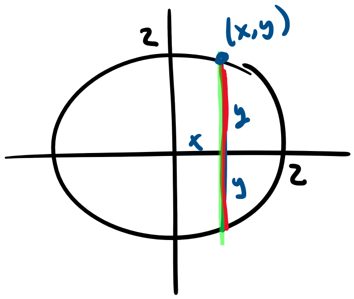
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x)\,\Delta x = \int_a^b A(x)\,dx$$

$x^2+y^2=4 \longrightarrow y^2=4-x^2$

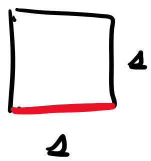
Example: The solid, S, has a base that is a circular disk with radius 2. Find the volume of the the solid if parallel cross sections taken perpendicular to the base are squares.

perp. to x-axis.

Base



Square



$s = 2y$

$A = s^2$

$A=(2y)^2=4y^2$

$A = 4(4-x^2)$

$= 16-4x^2$

1) draw base

2) draw cross section

3) find Area formula of cross section

4) dx or dy Integral

5) put Area formula in terms of the desired variable

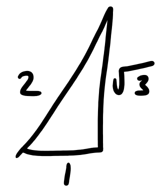
6) set up Integral

dx Integral.

$$V=\int_{-2}^{2} 16-4x^2 \, dx$$

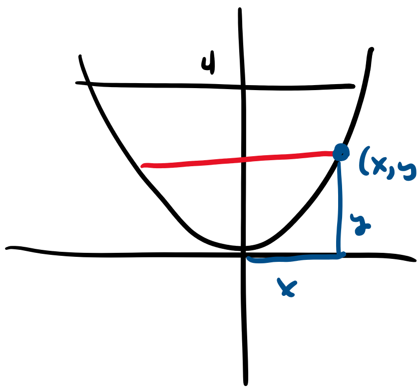
$$=\left(16x-\frac{4x^3}{3}\right)\Big|_{-2}^{2} = 32-\frac{32}{3}-\left(-32+\frac{32}{3}\right)$$

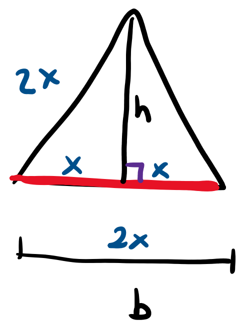
$$= 32-\frac{32}{3}+32-\frac{32}{3} = \frac{128}{3}$$



Example: The solid, S, has a base that is bounded by the equations: $y = x^2$ and $y = 4$. Find the volume of the solid if parallel cross sections are equilateral triangles that are perpendicular to the y-axis

dy Integral





$b = 2x$

$A = \frac{1}{2} b h$

$x^2 + h^2 = (2x)^2$

$h^2 = 4x^2 - x^2 = 3x^2$

$h = \sqrt{3}\, x$

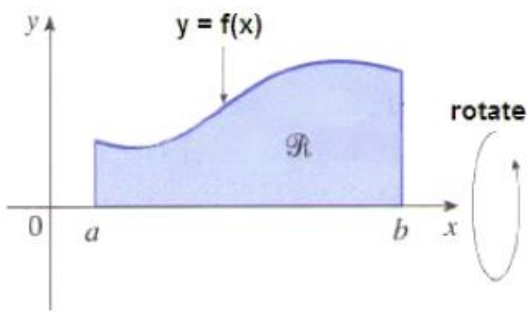
$A = \frac{1}{2}(2x)(\sqrt{3}\, x)$

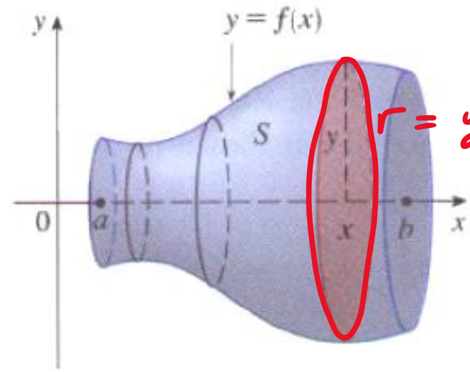
$A = \sqrt{3}\, x^2 = \sqrt{3}\, y$

$$V = \int_0^4 \sqrt{3}\, y \, dy = \sqrt{3} \left. \frac{y^2}{2} \right|_0^4 = \frac{16\sqrt{3}}{2} - 0 = 8\sqrt{3}$$

Page 4: disk method

Now lets consider rotating a region bounded between the x-axis and the function $f(x)$ from $x = a$ to $x = b$ around the x-axis.





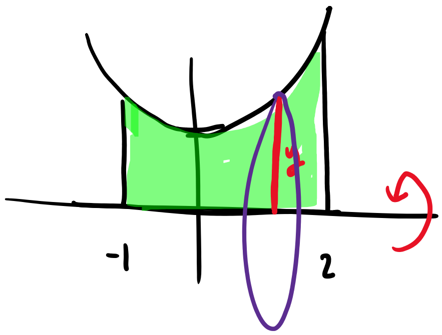
$r = y = f(x)$

$A = \pi r^2 = \pi (f(x))^2$

$$V = \int_a^b \pi (f(x))^2 \, dx$$

Example: Find the volume of the solid obtained by rotating the region bounded by the following around the x-axis.

$y = x^2 + 1$
x-axis
$x = -1$
$x = 2$



$A = \pi y^2$
$= \pi (x^2+1)^2$

$$V = \int_{-1}^{2} \pi (x^2+1)^2 \, dx$$

$$V = \pi \int_{-1}^{2} x^4 + 2x^2 + 1 \, dx = \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x \right]\Big|_{-1}^{2}$$

$$= \pi \left[\frac{32}{5} + \frac{16}{3} + 2 - \left(\frac{-1}{5} - \frac{2}{3} - 1 \right) \right]$$

$$= \pi \left[\frac{32}{5} + \frac{16}{3} + 2 + \frac{1}{5} + \frac{2}{3} + 1 \right]$$

$$= \pi \left[\frac{33}{5} + \frac{18}{3} + 2 + 1 \right] = \pi \left[\frac{33}{5} + 9 \right] = \pi \left(\frac{33+45}{5} \right)$$

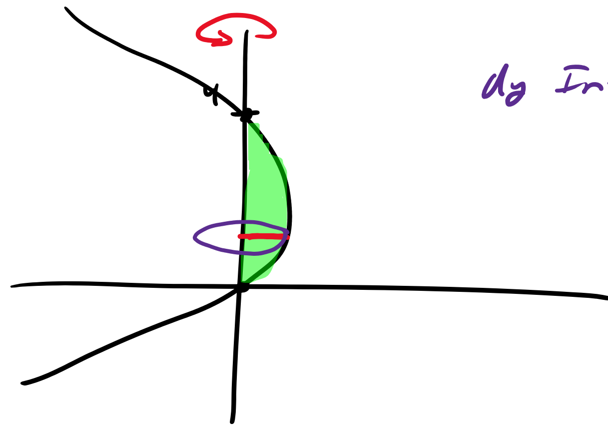
$$= \frac{78\pi}{5}$$

Example: Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

$x = 4y - y^2 = y(4-y)$
$x = 0$

$r = x$

dy Integral



$A = \pi r^2 = \pi x^2$

$A = \pi (4y - y^2)^2$

$$V = \int_0^4 \pi (4y - y^2)^2 \, dy = \dots = \frac{512\pi}{15}$$

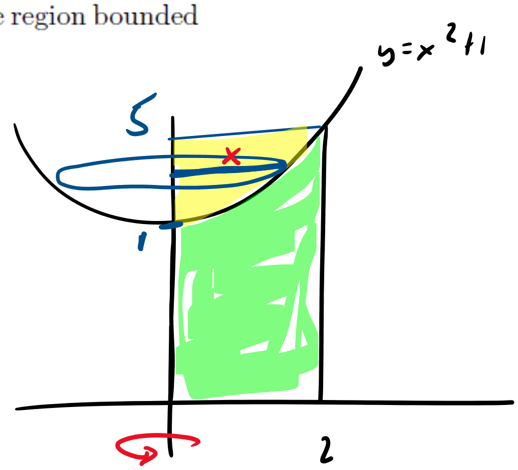
Example: Find the volume of the solid obtained by rotating the region bounded by the given curves around the y-axis.

$y = x^2 + 1$ $\quad x^2 = y-1$ $\quad x = \sqrt{y-1}$ $\qquad r = x$ $\quad r = \sqrt{y-1}$

$y = 0$

$x = 0$

$x = 2$



yellow + green Region

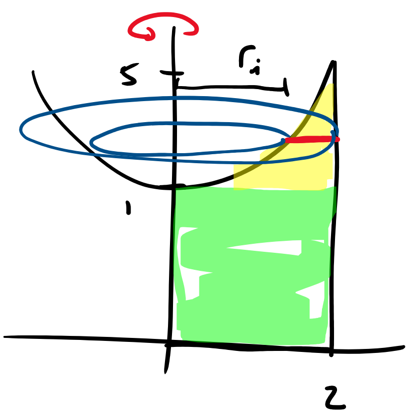
$V = \pi r^2 h = \pi (2)^2 (5)$

$= 20\pi$

$$\text{yellow volume} = \int_1^5 \pi r^2 \, dy = \int_1^5 \pi \left(\sqrt{y-1}\right)^2 dy$$

$$= \int_1^5 \pi (y-1) \, dy = \pi \left[\frac{y^2}{2} - y \right] \Big|_1^5 = \cdots = 12\pi$$

green volume $= 20\pi - 12\pi = 8\pi$



washer method. on $1 \le y \le 5$

$r_o = 2$

$y = x^2 + 1$

$x = \sqrt{y-1}$

$r_i = x = \sqrt{y-1}$

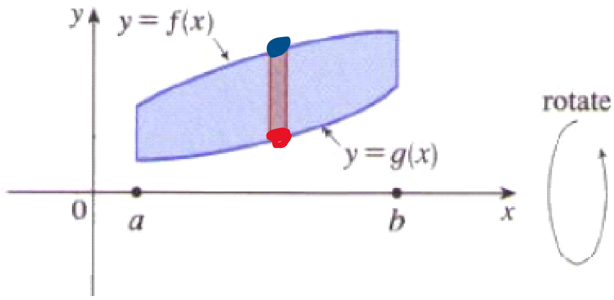
$$\underbrace{\begin{matrix} \pi r^2 h \\ \pi (2)^2 (1) \end{matrix}}_{\text{green part.}} + \overbrace{\int_1^5 \pi \left[(2)^2 - \left(\sqrt{y-1}\right)^2 \right] dy}^{\text{yellow part}}$$

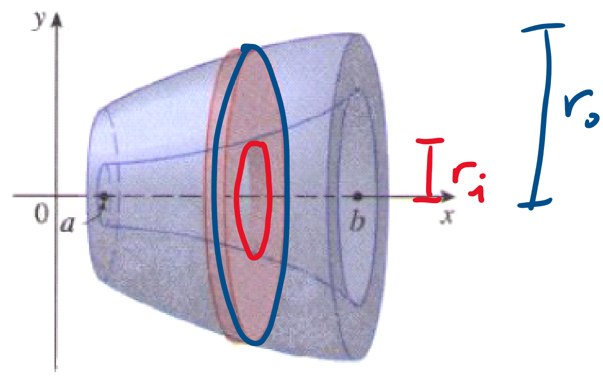
$$4\pi + \int^5 \pi \left[4 - (y-1) \right] dy$$

$$4\pi + \int_1^5 \pi\left[4 - (y-1)\right]\,dy$$

Page 8: washer method

Now lets consider rotating a region bounded between the function $f(x)$ and $g(x)$ from $x = a$ to $x = b$ around the x-axis.





$$\pi r_o^2 - \pi r_i^2 = \pi\left[r_o^2 - r_i^2\right]$$

$$V = \int_a^b \pi\left[r_o^2 - r_i^2\right] dx$$

Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around x-axis.

$y = x^2 + 2$

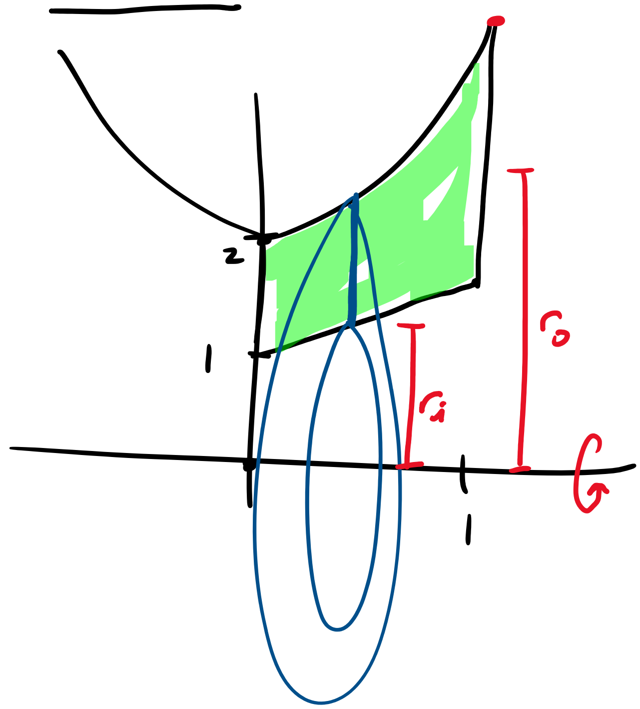
→ $2y - x = 2$

$x = 0$

$x = 1$

$2y = x + 2$

$y = \frac{1}{2}x + 1$



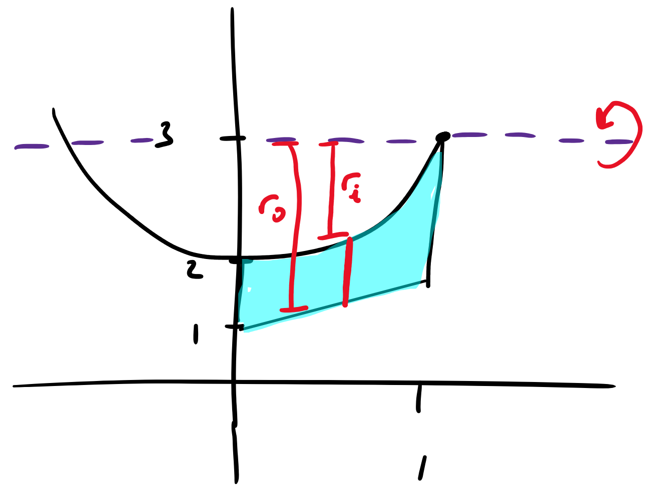
$r_o = y = x^2 + 2$

$r_i = y = \frac{1}{2}x + 1$

$$V = \int_0^1 \pi \left[(x^2+2)^2 - \left(\tfrac{1}{2}x + 1\right)^2 \right] dx = \cdots = \frac{79}{20}\pi$$

Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around $y = 3$

$y = x^2 + 2$
$2y - x = 2$ $\;\rightarrow\; y = \frac{1}{2}x + 1$
$x = 0$
$x = 1$



Top - Bottom

$$r_i = 3 - (x^2+2) = 3 - x^2 - 2 = 1 - x^2$$

$$r_o = 3 - \left(\tfrac{1}{2}x + 1\right) = 3 - \tfrac{1}{2}x - 1 = 2 - \tfrac{1}{2}x$$

washer

$$V = \int_0^1 \pi\left[\left(2 - \tfrac{1}{2}x\right)^2 - \left(1 - x^2\right)^2\right] dx$$

$$= \cdots = \frac{51}{20}\pi$$

Example: Set up the integral(s) that would give the volume of the solid obtained by rotating the region bounded by the given curves around $x = -3$.

$y = x^3$
$y = 2x + 4$
$x = 0$

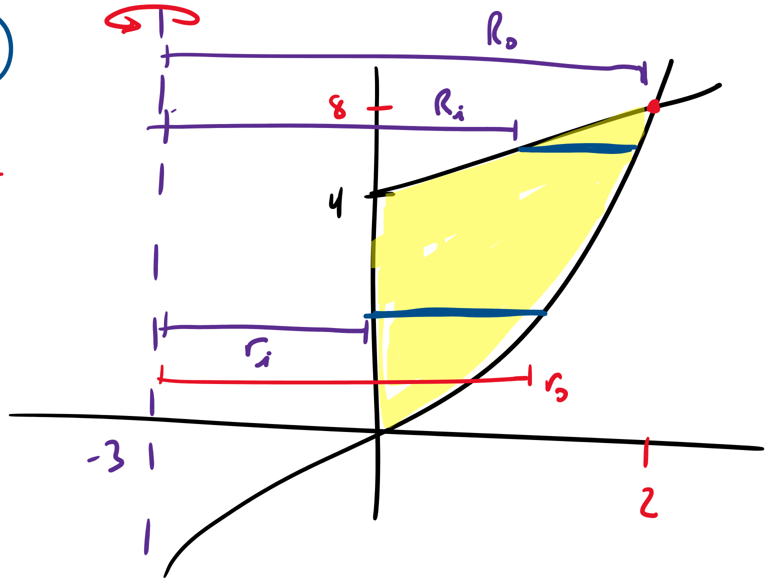
washer method

dy Integral

$x = \sqrt[3]{y}$

$2x = y - 4$

$x = \frac{1}{2}y - 2$



Top.

$R_o = \sqrt[3]{y} + 3$

$R_i = \frac{1}{2}y - 2 - (-3)$
$= \frac{1}{2}y - 2 + 3 = \frac{1}{2}y + 1$

Bottom part.

$r_i = 3 = 0 - (-3) = 3$

$r_o = x_{cubic} - (-3)$
$= \sqrt[3]{y} - (-3)$
$= \sqrt[3]{y} + 3$

Top | Bottom

$$\int_4^8 \pi\left[\left(\sqrt[3]{y}+3\right)^2 - \left(\frac{1}{2}y+1\right)^2\right]dy + \int_0^4 \pi\left[\left(\sqrt[3]{y}+3\right)^2 - \left(3\right)^2\right]dy$$

$$= \cdots = \frac{928}{15}\pi$$