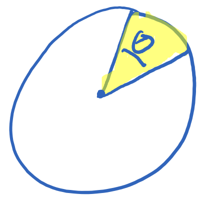
Section 10.4: Areas and Length in Polar Coordinates

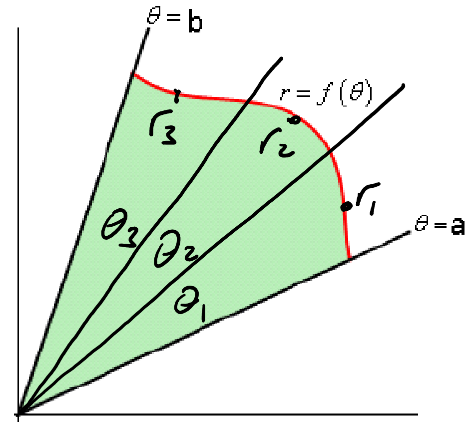
We would like to find the area of the region that is between the pole (origin) and the polar equation $r = f(\theta)$ from $\theta = a$ to $\theta = b$.

To be able to find this area we start back with the area of a circle being $A = \pi r^2$.

A sector of a circle, which is a part of the circle formed by the central angle θ, has an area that is proportional to the whole circle.



$A = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2}r^2\theta$



$\frac{1}{2}r_1^2\theta_1 + \frac{1}{2}r_2^2\theta_2 + \frac{1}{2}r_3^2\theta_3$

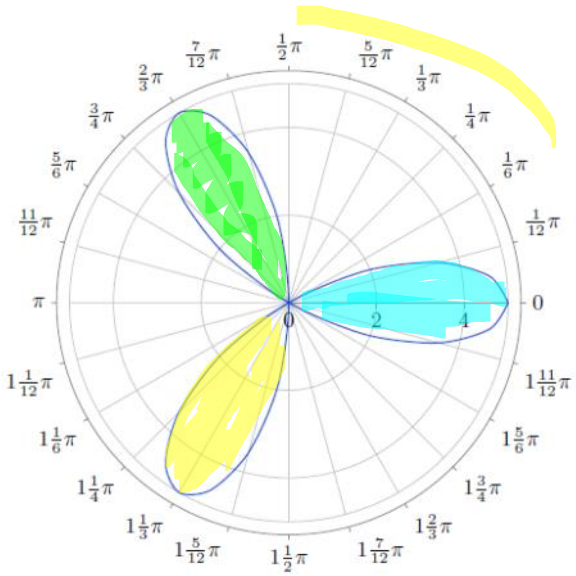
Now partition the region(on the right) where $\theta_1 = a$ to $\theta_n = b$. The area of each of the smaller sectors is given by $A_i = \frac{1}{2}r_i^2\Delta\theta$. Then area of the region is approximated by $A \approx \sum \frac{1}{2}r_i^2\Delta\theta$.

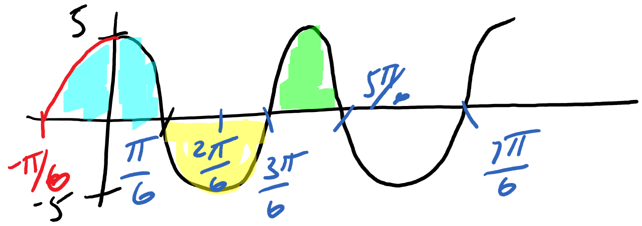
Thus the area of the region is $A = \int_a^b \frac{1}{2}r^2 d\theta$, where $r = f(\theta)$.

Pg 2

Wednesday, November 20, 2019 4:14 PM

Example: Find the area of one petal of the graph $r = 5\cos(3\theta)$.





$$3\theta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

Interval $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}\left(5\cos(3\theta)\right)^2 d\theta = \int_{-\pi/6}^{\pi/6} \frac{25}{2}\cos^2(3\theta)\, d\theta$$

$$= 2\int_0^{\pi/6} \frac{25}{2}\cos^2(3\theta)\, d\theta = 25\int_0^{\pi/6} \cos^2(3\theta)\, d\theta$$

$$= 25\int_0^{\pi/6} \frac{1}{2}\left[1 + \cos(2(3\theta))\right] d\theta$$

$$= \frac{25}{2}\int_0^{\pi/6} 1 + \cos(6\theta)\, d\theta$$

$$= \frac{25}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\right]_0^{\pi/6}$$

$$= \frac{25}{2}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin(\pi)\right) - \left(0 + \frac{1}{6}\sin(0)\right)\right]$$

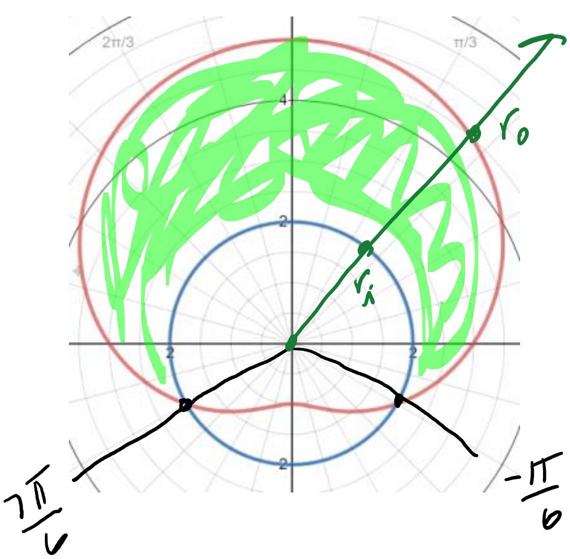
$$= \frac{25\pi}{12}$$

$A = \int \frac{1}{2} r^2 \, d\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Example: Find the area inside $r = 3 + 2\sin\theta$ and outside the circle $r = 2$.



$3 + 2\sin\theta = 2$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6} \quad \theta = -\frac{\pi}{6}$

$-\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$

$$A = \int_{-\frac{\pi}{6}}^{7\pi/6} \frac{1}{2}\left[(3 + 2\sin\theta)^2 - (2)^2\right] d\theta$$

$$= 2\int_{-\pi/6}^{\pi/2} \frac{1}{2}\left[9 + 12\sin\theta + 4\sin^2\theta - 4\right] d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\pi/2} 5 + 12\sin\theta + 4\sin^2\theta \quad d\theta$$

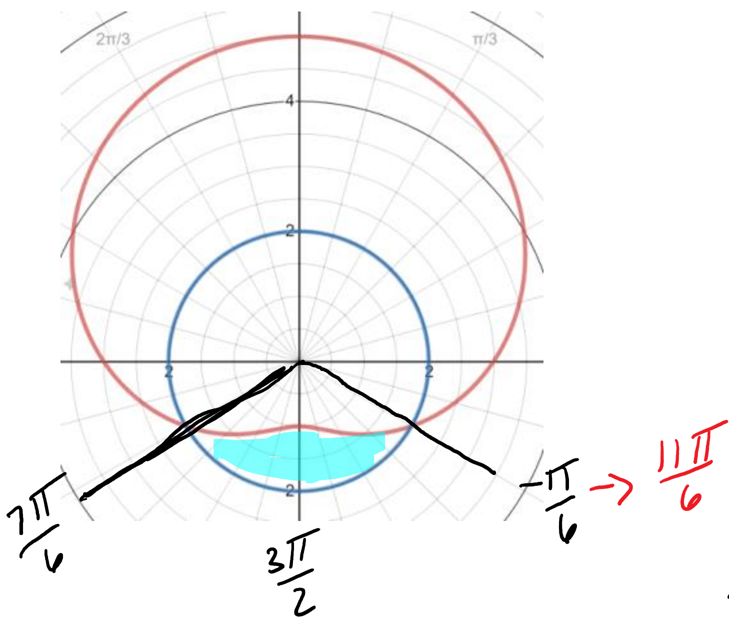
$$= \int_{\pi/}^{\pi/2} 5 + 12\sin\theta + 4 \cdot \frac{1}{2}\left[1 - \cos 2\theta\right] d\theta$$

$$= \int_{-\pi/6} 5 + 12\sin\theta + 4 \cdot \frac{1}{2}\left[\quad \right]$$

$$= \int_{-\pi/6}^{\pi/2} 5 + 12\sin\theta + 2 - 2\cos 2\theta \, d\theta$$

$$= \cdots = \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}$$

Example: Find the area inside the circle $r = 2$ and outside $r = 3 + 2\sin\theta$



$$\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$$

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left[2^2 - (3 + 2\sin\theta)^2\right] d\theta$$

$$A = 2\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} \frac{1}{2}\left[4 - (3 + 2\sin\theta)^2\right] d\theta$$

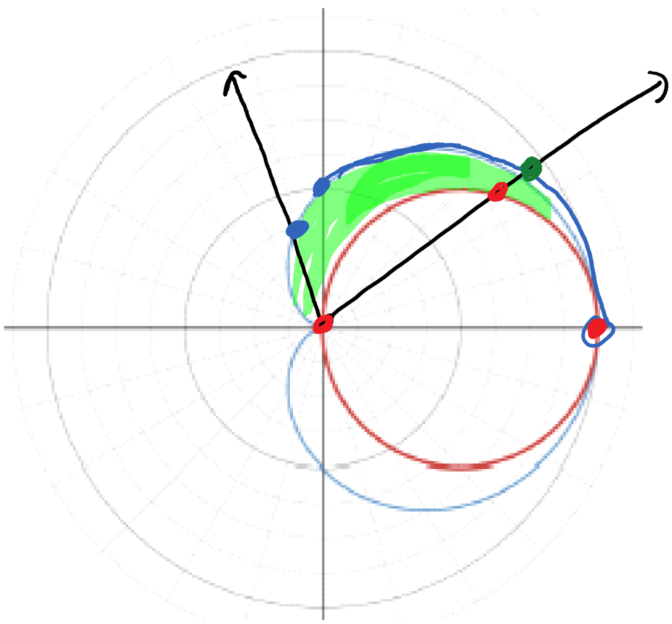
$$= \dots = \frac{11\sqrt{3}}{2} - \frac{7\pi}{3}$$

Pg 5

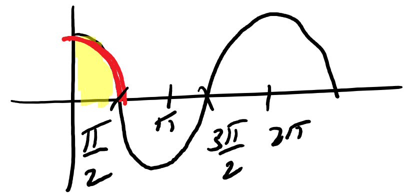
Wednesday, November 20, 2019 4:14 PM

$\int \frac{1}{2} r^2 \, d\theta$

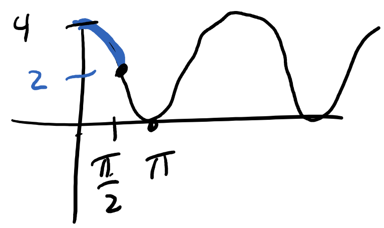
Example: Setup the integral(s) that give the area above the x-axis and inside $r = 2 + 2\cos\theta$ and outside $r = 4\cos\theta$



$r = 4\cos\theta$



$r = 2 + 2\cos\theta$



$0 \le \theta \le \frac{\pi}{2}$ $\qquad$ $\frac{\pi}{2} \le \theta \le \pi$

$$\int_0^{\pi/2} \frac{1}{2}\left[(2+2\cos\theta)^2 - (4\cos\theta)^2\right] d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}\left[2+2\cos\theta\right]^2 d\theta$$

Pg 6 arc length

Wednesday, November 20, 2019 4:14 PM

Arc Length

From section 10.2 we know the length of a curve is $L = \int_a^b ds$ where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$.

$r = f(\theta)$

Find the arc length of the polar curve $r =$ for $a \leq \theta \leq b$. Once again we assume that the curve is traced exactly once.

We start with $x = r\cos\theta$ and $y = r\sin\theta$ or $\boxed{x = f(\theta)\cos\theta \text{ and } y = f(\theta)\sin\theta}$
We know the formula for ds.

$$ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \cdots \text{lots of algebra} \cdots = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

Example: Find the length of the curve $r = \theta$ for $0 \le \theta \le 1$.

$$ds = \sqrt{r^2 + (r')^2}\, d\theta$$

$$L = \int_0^1 \sqrt{\theta^2 + (1)^2}\, d\theta = \int_0^1 \sqrt{\theta^2+1}\, d\theta$$

trig sub.

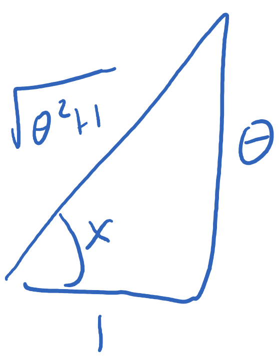
$\theta = \tan(x)$

$d\theta = \sec^2 x\, dx$

$$= \int_{\theta=0}^{\theta=1} \sqrt{\tan^2 x + 1}\, \sec^2(x)\, dx = \int_{\theta=0}^{\theta=1} \sqrt{\sec^2(x)} \cdot \sec^2(x)\, dx$$

$$= \int_{\theta=0}^{\theta=1} \sec^3 x\, dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\left|\sec(x) + \tan(x)\right| \Big|_{\theta=0}^{\theta=1}$$

$$= \frac{1}{2}\frac{\sqrt{\theta^2+1}}{1} \cdot \frac{\theta}{1} + \frac{1}{2}\ln\left|\sqrt{\theta^2+1} + \frac{\theta}{1}\right| \Big|_{\theta=0}^{\theta=1}$$



$$= \cdots = \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$

Example: Find the length of the curve $r = -4\sin\theta$ for $0 \le \theta \le \frac{2\pi}{3}$

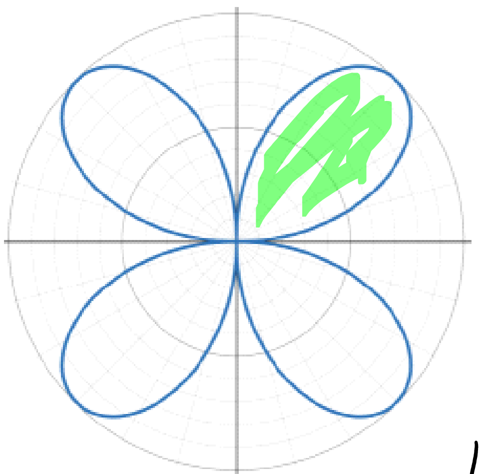
$$L = \int_0^{2\pi/3} \sqrt{(-4\sin\theta)^2 + (-4\cos\theta)^2}\, d\theta$$

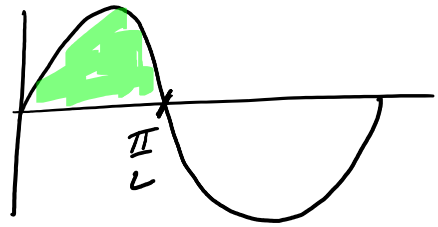
$$= \int_0^{2\pi/3} \sqrt{16\sin^2\theta + 16\cos^2\theta}\, d\theta = \int_0^{2\pi/3} \sqrt{16}\, d\theta$$

$$= \int_0^{2\pi/3} 4\, d\theta = 4\theta \Big|_0^{2\pi/3} = \boxed{\frac{8\pi}{3}}$$

$2\theta = \pi \to \theta = \frac{\pi}{2}$

Example: Setup the integral that would give the length of the curve that forms one of the loops for $r = \sin(2\theta)$.





$0 \le \theta \le \frac{\pi}{2}$

$$L = \int_0^{\pi/2} \sqrt{(\sin(2\theta))^2 + (2\cos(2\theta))^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + 4\cos^2(2\theta)}\, d\theta$$