Section 11.10: Taylor and Maclaurin Series

Definition: If a function has a power series representation, then this power series is referred to as the Taylor series of the function f at a (or about a or centered at a). If this series is centered at $x = 0$, then this series is given the special name Maclaurin series.

Theorem: If $f(x)$ has a power series representation at a, i.e. centered at $x = a$, that is if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with $|x - a| < R$ then its coefficients are given by the formula

Radius of convergence $= R$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Very Important

you must know this.

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$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

$f(a) = C_0 = \frac{f^{(0)}(a)}{0!}$

$$f'(x) = c_1 + 2 * c_2(x-a) + 3 * c_3(x-a)^2 + 4 * c_4(x-a)^3 + 5 * c_5(x-a)^4 + \dots$$

$f'(a) = C_1$ $\quad C_1 = \frac{f'(a)}{1!}$

$$f''(x) = 2 * 1 * c_2 + 3 * 2 * c_3(x-a) + 4 * 3 * c_4(x-a)^2 + 5 * 4 * c_5(x-a)^3 + \dots$$

$f''(a) = 2 \cdot 1 \cdot C_2$ $\quad C_2 = \frac{f''(a)}{2!}$

$$f'''(x) = 3 * 2 * 1 * c_3 + 4 * 3 * 2 * c_4(x-a) + 5 * 4 * 3 * c_5(x-a)^2 + 6 * 5 * 4 * c_6(x-a)^3 \dots$$

$f'''(a) = 3 \cdot 2 \cdot 1 \cdot C_3 = 3!\, C_3$ $\quad C_3 = \frac{f^{(3)}(a)}{3!}$

$$f^{(4)}(x) = 4 * 3 * 2 * 1 * c_4 + 5 * 4 * 3 * 2 * c_5(x-a) + 6 * 5 * 4 * 3 * c_6(x-a)^2 + \dots$$

$f^{(4)}(a) = 4 \cdot 3 \cdot 2 \cdot 1 \cdot C_4 = 4!\, C_4$ $\quad C_4 = \frac{f^{(4)}(a)}{4!}$

$$f^{(5)}(x) = 5 * 4 * 3 * 2 * 1 * c_5 + 6 * 5 * 4 * 3 * 2 * c_6(x-a) + \dots$$

$f^{(5)}(a) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\, C_5 = 5!\, C_5$ $\quad C_5 = \frac{f^{(5)}(a)}{5!}$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = e^{2x}$

↳ centered at $x = 0$ $(a = 0)$

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$n=0$ ↳ $f(x) = e^{2x}$

$n=1$ $f'(x) = 2e^{2x}$

$n=2$ $f''(x) = 2^2 e^{2x}$

$f'''(x) = 2^3 e^{2x}$

$f^{(4)}(x) = 2^4 e^{2x}$

$$f^{(n)}(x) = 2^n e^{2x}$$

Check to see if it works for all $n \geq 0$ ✓

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$c_n = \frac{2^n e^0}{n!} = \frac{2^n}{n!}$$

$$e^{2x} = \sum_{n=0} c_n (x-0)^n = \sum_{n=0} \frac{2^n}{n!} x^n = \sum_{n=0} \frac{2^n x^n}{n!}$$

use Ratio test can show Radius of conv.

$R = \infty$

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Example: Find the Maclaurin series and the radius of convergence for $f(x) = \sin(2x)$

$a = 0$

$f^{(n)}(x)$

$f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$f''(x) = -2^2 \sin(2x)$

$f'''(x) = -2^3 \cos(2x)$

$f^{(4)} = 2^4 \sin(2x)$

$f^{(5)} = 2^5 \cos(2x)$

$f(0) = 0$

$f'(0) = 2$

$f''(0) = 0$

$f'''(0) = -2^3$

$f^{(4)}(0) = 0$

$f^{(5)}(0) = 2^5$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -2^7$

$f^{(8)}(0) = 0$

$f^{(9)}(0) = 2^9$

$$C_n = \frac{f^{(n)}(0)}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

$$= \underbrace{\frac{f^{(0)}(0)}{0!}}_{n=0} + \underbrace{\frac{f'(0)}{1!}(x-0)}_{n=1} + \underbrace{\frac{f''(0)}{2!}(x-0)^2}_{n=2} + \underbrace{\frac{f'''(0)}{3!}(x-0)^3}_{n=3} + \underbrace{\frac{f^{(4)}}{4!}(x-0)^4}_{n=4} + \ldots\ldots$$

$$f(x) = 0 + \frac{2}{1!}x + 0 - \frac{2^3}{3!}x^3 + 0 + \frac{2^5}{5!}x^5 + 0 - \frac{2^7}{7!}x^7 + \ldots\ldots$$

$$f(x) = \underbrace{\frac{2^1}{1!}x}_{n=0} - \underbrace{\frac{2^3}{3!}x^3}_{n=1} + \underbrace{\frac{2^5}{5!}x^5}_{n=2} - \underbrace{\frac{2^7}{7!}x^7}_{n=3} + \underbrace{\frac{2^9}{9!}x^9}_{n=4} - \underbrace{\frac{2^{11}}{11!}x^{11}}_{n=5} + \ldots\ldots$$

$$\boxed{\sin(2x) = \sum_{n=0} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}}$$

use Ratio test to show $R = \infty$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \cos(2x)$

$R = \infty$

$$g(x) = \sin(2x) = \sum_{n=0} \frac{(-1)^n \, 2^{2n+1} \, x^{2n+1}}{(2n+1)!}$$

$$g(x) = \sin(2x)$$

$$g'(x) = 2\cos(2x)$$

$$\cos(2x) = \frac{1}{2} g'(x)$$

$$= \frac{1}{2} \sum_{n=0} \frac{(-1)^n \, 2^{2n+1} \, (2n+1) x^{2n}}{(2n+1)!}$$

$$(2n+1)! = (2n+1)(2n)!$$

$$\boxed{\cos(2x) = \sum_{n=0} \frac{(-1)^n \, 2^{2n} \, x^{2n}}{(2n)!}}$$

$$2n! \neq (2n)!$$
$$2(n!)$$

$R = \infty$

Important Maclaurin series

Note: These are the only building blocks that you do not have to prove the derivation. Any other "building blocks" used, must be proved.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots \quad R = \infty$$

$$e^{2x} = \sum_{n=0} \frac{(2x)^n}{n!} = \sum_{n=0} \frac{2^n x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad R = \infty$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad R = \infty$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad |x| \le 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad -1 < x \le 1$$

$|x| < 1$

Example: Find the Maclaurin series and the radius of convergence for

A) $f(x) = \sin(3x)$ $= \sum_{n=0} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$

$R = \infty$

B) $f(x) = x^2 e^{5x}$ $= x^2 \sum_{n=0} \frac{(5x)^n}{n!} = x^2 \sum_{n=0} \frac{5^n x^n}{n!}$

$= \sum_{n=0} \frac{5^n x^{n+2}}{n!}$

$R = \infty$

Example: Find the Maclaurin series and the radius of convergence for $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

$$f(x) = \ln(1+x) - \ln(1-x)$$

$$= \ln(1+x) - \ln(1+\boxed{(-x)})$$

$|-x| < 1$

$|x| < 1$

$R = 1$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{(-1)^n (-x)^{n+1}}{n+1}$$

$|x| < 1$

$R = 1$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{\boxed{(-1)^n (-1)^{n+1}} x^{n+1}}{n+1}$$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1}}{n+1} - \sum_{n=0} \frac{-x^{n+1}}{n+1}$$

$$= \sum_{n=0} \frac{(-1)^n x^{n+1} + x^{n+1}}{n+1} = \sum_{n=0} \left(\frac{(-1)^n + 1}{n+1}\right) x^{n+1}$$

$R = 1$

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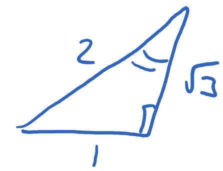
$a = \frac{\pi}{6}$

Example: Find the Taylor series of $f(x) = \sin(x)$ at $x = \frac{\pi}{6}$ $\rightarrow 30°$



$f(x) = \sin(x)$ $\qquad f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos(x)$ $\qquad f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin(x)$ $\qquad f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos(x)$ $\qquad f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \sin(x)$ $\qquad f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$C_n = \frac{f^{(n)}(a)}{n!}$$

$$C_n = \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}$$

$$\sin(x) = C_0 + C_1\left(x - \frac{\pi}{6}\right) + C_2\left(x - \frac{\pi}{6}\right)^2 + C_3\left(x - \frac{\pi}{6}\right)^3 + \cdots$$

$$= \frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{1!}\left(x - \frac{\pi}{6}\right) + \frac{-\frac{1}{2}}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{-\frac{\sqrt{3}}{2}}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{\frac{1}{2}}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots$$

This is a series that we can not find Sigma notation.

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Example: Find the Taylor series of $f(x) = \dfrac{1}{x^2}$ about $a = 3$

$n=0$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$f'(x) = -2x^{-3} = \frac{-2}{x^3}$$

$$f''(x) = 2\cdot 3x^{-4} = \frac{2\cdot 3}{x^4}$$

$$f'''(x) = -2\cdot 3\cdot 4x^{-5} = \frac{-2\cdot 3\cdot 4}{x^5}$$

$$f^{(4)}(x) = 2\cdot 3\cdot 4\cdot 5x^{-6} = \frac{2\cdot 3\cdot 4\cdot 5}{x^6}$$

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}}$$

does it work for all n. i.e, $n \geq 0$ yes :)

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$c_n = \frac{f^{(n)}(3)}{n!} = \frac{1}{n!}\,\frac{(-1)^n (n+1)!}{(3)^{n+2}} = \frac{(-1)^n (n+1)}{3^{n+2}}$$

$$\boxed{\sum_{n=0} c_n (x-a)^n}$$

$$\boxed{\sum_{n=0} \frac{(-1)^n (n+1)}{3^{n+2}} (x-3)^n = \frac{1}{x^2}}$$

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$0! = 1$

Example: Find the Taylor series of $f(x) = \ln(x)$ about $a = 2$

$n=0$

$f(x) = \ln(x)$

$f' = \frac{1}{x} = x^{-1}$

$f'' = -x^{-2}$

$f''' = 2x^{-3}$

$f^{(4)} = -2\cdot 3x^{-4}$

$f^{(5)} = 2\cdot 3\cdot 4x^{-5}$

$$f^{(n)} = \frac{(-1)^{n+1}(n-1)!}{\boxed{x}^n}$$

does this work for all n values?

yes if $n \geq 1$

$\boxed{\text{no if } n=0}$

$$c_n = \frac{f^{(n)}(\boxed{2})}{n!} = \frac{1}{n!} \cdot \frac{(-1)^{n+1}(n-1)!}{2^n}$$

$$= \frac{(-1)^{n+1}\cancel{(n-1)!}}{n\cdot\cancel{(n-1)!}\,2^n} = \frac{(-1)^{n+1}}{n2^n} \quad \boxed{\text{if } n \neq 0}$$

$$\ln(x) = \sum_{n=0} c_n(x-2)^n = c_0 + \sum_{n=1} c_n(x-2)^n$$

$$\boxed{\ln(x) = \ln(2) + \sum_{n=1} \frac{(-1)^{n+1}}{n2^n}(x-2)^n}$$

$$c_0 = \frac{f^{(0)}(2)}{0!} = f(2) = \ln(2)$$

$f(x) = \ln(x)$

Example: Find the Taylor series of $f(x) = \frac{1}{\sqrt{x}}$ about $a = 4$

$n=0$

$\hookrightarrow f(x) = x^{-1/2} = \frac{1}{\sqrt{x}}$

$f'(x) = -\frac{1}{2} x^{-3/2}$

$f''(x) = \frac{1}{2} \cdot \frac{3}{2} x^{-5/2}$

$f'''(x) = -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} x^{-7/2}$

$f^{(4)}(x) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} x^{-9/2}$

$$f^{(n)}(x) = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \, x^{\frac{2n+1}{2}}}$$

does it work for all n?

does not work for $n=0$.

$$C_n = \frac{f^{(n)}(4)}{n!} = \frac{1}{n!} \cdot \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \, 4^{\frac{2n+1}{2}}}$$

$$= \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! \, 2^n \, 2^{2n+1}} = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! \, 2^{3n+1}} = C_n$$

as $n \neq 0$

$$\frac{1}{\sqrt{x}} = \sum_{n=0} C_n (x-4)^n = C_0 + \sum_{n=1} C_n (x-4)^n$$

$$\boxed{\frac{1}{\sqrt{x}} = \frac{1}{2} + \sum_{n=1} \frac{(-1)^n \, 1 \cdot 3 \cdot 5 \cdot \cdots (2n-1)}{n! \, 2^{3n+1}} (x-4)^n}$$

$$c_0 = \frac{f^{(0)}(4)}{0!} = f(4) = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

Example: If $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n4^n}$, find $f^{(48)}(3)$.

$$\sum_{n=0} c_n (x-a)^n$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$c_n = \boxed{\frac{1}{n4^n} = \frac{f^{(n)}(3)}{n!}}$$

$$f^{(n)}(3) = \frac{n!}{n4^n} = \frac{n\cdot(n-1)!}{n4^n} = \frac{(n-1)!}{4^n}$$

$$\boxed{f^{(48)}(3) = \frac{(48-1)!}{4^{48}} = \frac{47!}{4^{48}}}$$

$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

Example: Use series to evaluate this integral.

$$\int \frac{e^{x^2} - 1 - x^2}{x} dx$$

$n=0$ $n=1$ $n=2$

$$e^{x^2} = \sum_{n=0} \frac{(x^2)^n}{n!} = \sum_{n=0} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$$e^{x^2} - 1 - x^2 = \sum_{n=2}^{\infty} \frac{x^{2n}}{n!}$$

$$\frac{1}{x}\left(e^{x^2} - 1 - x^2\right) = \frac{1}{x} \sum_{n=2} \frac{x^{2n}}{n!} = \underbrace{\sum_{n=2} \frac{x^{2n-1}}{n!}}_{} = \frac{e^{x^2} - 1 - x^2}{x}$$

$$\int \frac{e^{x^2} - 1 - x^2}{x} dx = \int \sum_{n=2} \frac{x^{2n-1}}{n!} dx = C + \underbrace{\sum_{n=2} \frac{x^{2n}}{(2n) \cdot n!}}_{}$$

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$$\cos(x) = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

Example: Find the sum of these series.

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$

$$= \sum_{n=0} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \cos(60)$$

$$= \frac{1}{2}$$

$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

(B) $\displaystyle\sum_{n=2}^{\infty} \frac{5^n x^{3n}}{n!} = \sum_{n=2} \frac{(5x^3)^n}{n!}$

$$5^n x^{3n} = 5^n \cdot (x^3)^n = (5x^3)^n$$

missing $n=0$ + $n=1$ terms

$$= e^{5x^3} - 1 - \frac{5x^3}{1!}$$

$$= \boxed{e^{5x^3} - 1 - 5x^3}$$