**Definition:** Given a sequence $\{a_i\}$, we can construct an **infinite series** or **series** by adding the terms of the sequence. $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots$

**Definition:** The *nth partial sum of a* series, denoted $s_n$, is the sum of the first n-terms.

**NOTE: If the index starts at $i = 1$ then**

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

$s_1 = a_1$
$s_2 = s_1 + a_2 = a_1 + a_2$
$s_3 = s_2 + a_3 = a_1 + a_2 + a_3$
$s_4 = s_3 + a_4 = a_1 + a_2 + a_3 + a_4$
$s_5 = s_4 + a_5 = a_1 + a_2 + a_3 + a_4 + a_5$

Example: Find the $s_4$ for the series: $\sum_{i=4}^{\infty} \frac{1}{(i-2)^2} = \frac{1}{(4-2)^2} + \frac{1}{(5-2)^2} + \ldots$

$$= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \ldots$$

$$\boxed{S_4 = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}}$$

**How To Shift a Series:**

Example: Adjust the series $\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{2i}$ so that the index will now start at i=1.

$j = i - 2$

$j + 2 = i$

$$\sum_{j=1}^{\infty} 10\left(\frac{1}{3}\right)^{2(j+2)} = \sum_{j=1}^{\infty} 10\left(\frac{1}{3}\right)^{2j+4}$$

$$\boxed{\sum_{i=1}^{\infty} 10\left(\frac{1}{3}\right)^{2i+4}}$$

**Definition:** Let $\sum_{i=1}^{\infty} a_i$ be a series with $s_n$ being the $n$th partial sum of this series. If the **sequence of partial sums** $\{s_n\}$ converges to $s$, i.e. $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ converges to $s$ or that the series has a sum of $s$, $\sum_{i=1}^{\infty} a_i = s$. If $\{s_n\}$ does not converge, then the series $\sum_{i=1}^{\infty} a_i$ is said to be divergent.

$S_n \to 50$ as $n \to \infty$

| n | $a_n$ |
|---|---|
| 1 | 40 |
| 2 | 8 |
| 3 | 8/5 = 1.6 |
| 4 | 8/25 = 0.32 |
| 5 | 8/125 = 0.064 |
| 6 | 8/625 = 0.0128 |
| 7 | 8/ 3125 = 0.00256 |
| 8 | 8/15625 = 0.000512 |
| 9 | 8/78125 = 0.0001024 |
| 10 | 8/390625 = 0.00002048 |

| n | $s_n$ |
|---|---|
| 1 | 40 |
| 2 | 48 |
| 3 | 49.6 $= S_2 + a_3$ |
| 4 | 49.92 |
| 5 | 49.984 |
| 6 | 49.9968 |
| 7 | 49.99936 |
| 8 | 49.999872 |
| 9 | 49.9999744 |
| 10 | 49.99999488 |

Thus

$$\sum_{i=1}^{\infty} a_i = 50$$

**Theorem:** If the series $\sum_{i=1}^{\infty} a_i$ is convergent, then $\lim_{i\to\infty} a_i = 0$

Test for Divergence: If $\lim_{i\to\infty} a_i \neq 0$ or DNE, then the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example: Which of these series DO NOT have a chance at being convergent?

A) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$a_n = \frac{1}{n^3}$ $\quad \lim_{n\to\infty} \frac{1}{n^3} = 0$

may or may not conv.

B) $\sum_{n=1}^{\infty} \frac{3n+5}{7-2n}$

$\lim_{n\to\infty} \frac{3n+5}{7-2n} \overset{L'H}{=} \frac{3}{-2} = -\frac{3}{2} \neq 0$

by the test for div. The series will div.

C) $\sum_{n=1}^{\infty} \cos(e^{-n})$

$\lim_{n\to\infty} \cos(e^{-n}) = \cos(0) = 1$

by the test for div. this series will div.

Example: The series $\sum_{i=1}^{\infty} a_i$ has a $n$th partial sum given by $s_n$. Will the series converge or diverge? Find the formula for the $a_n$ term.

$$s_n = \frac{3n+5}{7-2n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3n+5}{7-2n} = \boxed{-\frac{3}{2}}$$

$$\sum_{i=1}^{\infty} a_i = -\frac{3}{2}$$ The series converges.

$$S_n = a_n + S_{n-1}$$

$$S_n - S_{n-1} = a_n$$

$$a_n = S_n - S_{n-1}$$

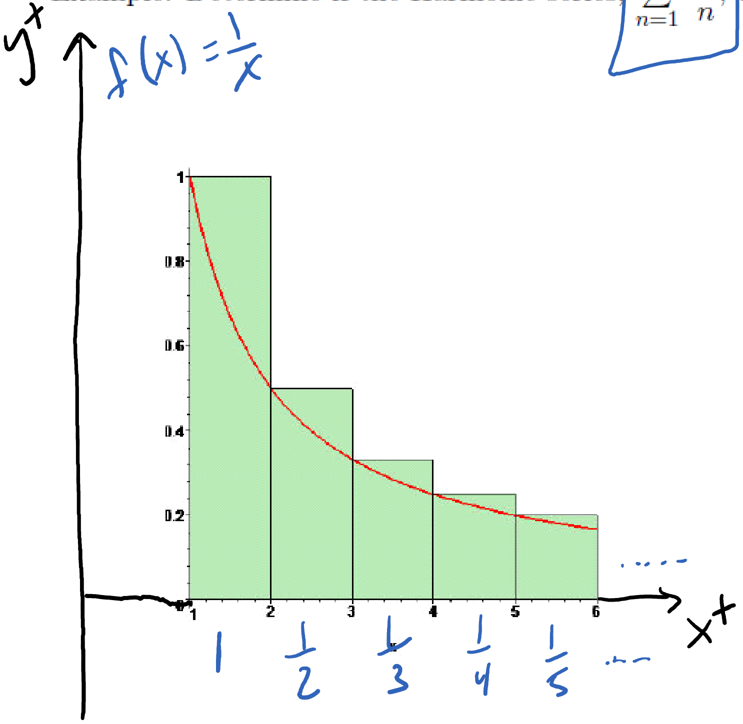
$$a_n = \frac{3n+5}{7-2n} - \frac{3(n-1)+5}{7-2(n-1)}$$

Find $a_4 = S_4 - S_3 = \frac{17}{-1} - \frac{14}{1} = -17 - 14 = -31$

$$\boxed{a_4 = -31}$$

Example: Determine if the Harmonic series, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$, converges or diverges.

$\sum_{i=1}^{\infty} \frac{1}{i}$




$$\underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{\text{p-integral}} < \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Rectangles}}$$

p = 1
div.

diverges

Example: The **geometric series** may be defined in a variety of methods.

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ....$$

$$\sum_{n=0}^{\infty} ar^{n} = a + ar + ar^2 + ar^3 + ....$$

$$\sum_{n=7}^{\infty} ar^{n-7} = a + ar + ar^2 + ar^3 + ....$$

conv. $\frac{a}{1-r}$ if $|r|<1$

**Proof of the Geometric Series:**
Consider the parital sum of the first n terms.

$$S_n = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

Multiply $S_n$ by $r$ to get: $rS_n = ar + ar^2 + ar^3 + ... + ar^n$

Now compute $S_n - rS_n$ and then solve for $S_n$

$$S_n - rS_n = a - ar^n$$

$$(1-r)S_n = a - ar^n$$

$$\boxed{S_n = \frac{a - ar^n}{1-r}}$$

$-1 < r < 1$

$$\text{Sum}= \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

$\to j$ $\to k$

**Theorem:** If $\sum a_n$ and $\sum b_n$ are covergent series, then so are the following series

$\sum ca_n = c\sum a_n$ (where c is a constant)

$\sum(a_n + b_n) = \sum a_n + \sum b_n \longrightarrow j + k$

$\sum(a_n - b_n) = \sum a_n - \sum b_n$

Example: Determine if these series are convergent or divergent. If the series is convergent, then give the sum of the series.

A) $1 - \frac{4}{3} + \frac{16}{9} - \frac{64}{27} + ....$

$1 + \left(-\frac{4}{3}\right) + \left(-\frac{4}{3}\right)^2 + \left(-\frac{4}{3}\right)^3 + \ldots$

$r = -\frac{4}{3}$ $a = 1$

Since $|r| \geq 1$

The series div.

B) $\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{i-1}$

wrong.

$a = 10 \quad r = \frac{1}{3} \qquad \text{Sum} = \frac{a}{1-r} = \frac{10}{1-\frac{1}{3}} = \frac{10}{\frac{2}{3}} = 10 \cdot \frac{3}{2} = 15$

$$\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{i-1} = \underbrace{10\left(\frac{1}{3}\right)^{2}}_{a} + \underbrace{10\left(\frac{1}{3}\right)^{3}}_{ar} + \underbrace{10\left(\frac{1}{3}\right)^{4}}_{ar^2} + \cdots$$

$$a = \frac{10}{9} \qquad r = \frac{1}{3} \qquad \text{Sum} = \frac{a}{1-r} = \frac{10/9}{1-\frac{1}{3}} = \frac{\frac{10}{9}}{\frac{2}{3}} = \frac{10}{9}\cdot\frac{3}{2} = \frac{5}{3}$$

The series converges to the sum of $\frac{5}{3}$.

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$$\sum_{i=1} 10\left(\frac{1}{3}\right)^{i-1} = 10 + 10\left(\frac{1}{3}\right) + \underbrace{10\left(\frac{1}{3}\right)^{2} + \cdots}$$

$$\underbrace{\sum_{i=1} 10\left(\frac{1}{3}\right)^{i-1}} = 10 + 10\left(\frac{1}{3}\right) + \sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{i-1}$$

$$15 = \underbrace{10 + \frac{10}{3}} + \underbrace{\sum_{i=3}^{\infty} 10\left(\frac{1}{3}\right)^{i-1}}$$

$$\sum_{i=3} 10\left(\frac{1}{3}\right)^{i-1} = 15 - 10 - \frac{10}{3} = 5 - \frac{10}{3}$$

$$= \frac{15}{3} - \frac{10}{3} = \frac{5}{3}$$

C) $\sum_{n=0}^{\infty} 7 * 4^{-n} 3^{n-1} = \sum_{n=0} 7 \cdot \frac{3^{n-1}}{4^n}$

$$= \underbrace{7 \cdot \frac{3^{-1}}{4^0}}_{a} + \underbrace{7 \cdot \frac{3^0}{4^1}}_{ar} + 7 \cdot \frac{3^1}{4^2} + 7 \cdot \frac{3^2}{4^3} + \cdots$$

$a = \frac{7 \cdot 3^{-1}}{4^0} = \frac{7}{3}$ $\quad r = \frac{3}{4}$ $\quad \frac{ar}{a} = r$

Since $|r| < 1$ the series conv.

$$\text{Sum} = \frac{a}{1-r} = \frac{7/3}{1 - 3/4} = \frac{7/3}{1/4} = \frac{28}{3}$$

D) $\sum_{i=1}^{\infty} \ln\left(\frac{i}{i+1}\right)$

Test for div.

$\lim_{i \to \infty} \ln\left(\boxed{\frac{i}{i+1}}\right) = \ln(1) = 0$

may or may not conv.

need a formula for the partial sum.

$S_n = \sum_{i=1}^{n} \ln\left(\frac{i}{i+1}\right) = \sum_{i=1}^{n} \left[\ln(i) - \ln(i+1)\right]$

$S_n = \ln(1) - \ln(2)$ $\quad i=1$

$+ \ln(2) - \ln(3)$ $\quad i=2$

$+ \ln(3) - \ln(4)$ $\quad i=3$

$+ \ln(4) - \ln(5)$ $\quad i=4$

$\vdots$ $\ldots$

$+ \ln(n-3) - \ln(n-2)$ $\quad i=n-3$

$+ \ln(n-2) - \ln(n-1)$ $\quad i=n-2$

$+ \ln(n-1) - \ln(n)$ $\quad i=n-1$

$+ \ln(n) - \ln(n+1)$ $\quad i=n$

Telescoping Series

$S_n = \ln(1) - \ln(n+1) = -\ln(n+1)$

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} -\ln(n+1) = -\infty$

Sequence of partial sums does not conv.

Thus the series will diverge

E) $\sum_{i=3}^{\infty}\left(\frac{1}{i-2}-\frac{1}{i}\right) = \sum_{i=3} \frac{2}{i^2-2i}$

partial sum $= 1 - \cancel{\frac{1}{3}}$ $\quad i=3$

$+ \frac{1}{2} - \cancel{\frac{1}{4}}$ $\quad i=4$

$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}$ $\quad i=5$

$+ \cancel{\frac{1}{4}} - \frac{1}{6}$ $\quad i=6$

$+ \cancel{\frac{1}{5}} - \frac{1}{7}$ $\quad i=7$

$+ \frac{1}{n-5} - \cancel{\frac{1}{n-3}}$ $\quad i=n-3$

$+ \frac{1}{n-4} - \cancel{\frac{1}{n-2}}$ $\quad i=n-2$

$+ \cancel{\frac{1}{n-3}} - \frac{1}{n-1}$ $\quad i=n-1$

$+ \cancel{\frac{1}{n-2}} - \frac{1}{n}$ $\quad i=n$

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partial sum $= 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$

$\lim_{n\to\infty}$ partial sum $= 1 + \frac{1}{2} = \frac{3}{2}$

The series converges. The sum of the series is $3/2$

Telescoping

F) $\sum_{i=1}^{\infty} e^{5/(i+1)} - e^{5/i}$

$$S_n = e^{5/2} - e^{5} \quad i=1$$
$$+ e^{5/3} - e^{5/2} \quad i=2$$
$$+ e^{5/4} - e^{5/3} \quad i=3$$
$$+ e^{5/n-1} - e^{5/n-2} \quad i=n-2$$
$$+ e^{5/n} - e^{5/n-1} \quad i=n-1$$
$$+ e^{5/n+1} - e^{5/n} \quad i=n$$

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$$S_n = e^{5/n+1} - e^{5}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(e^{\frac{5}{n+1}} - e^{5}\right) = e^{0} - e^{5}$$
$$= 1 - e^{5}$$

The series will converge and Its sum is $1-e^{5}$

Example: Use a geometric series to express $0.\overline{14}$ as a ratio of integers.

$$.\overline{14} = .14141414\ldots$$

$$= .14 + .0014 + .000014 + .00000014 + \ldots$$

$$= \underbrace{\frac{14}{100}}_{a} + \underbrace{\frac{14}{100}\cdot\frac{1}{100}}_{ar} + \frac{14}{100}\cdot\left(\frac{1}{100}\right)^2 + \frac{14}{100}\left(\frac{1}{100}\right)^3 + \ldots$$

$$r = \frac{1}{100}$$

$|r|<1$

Conv.

$$S_{num} = \frac{a}{1-r} = \frac{\frac{14}{100}}{1-\frac{1}{100}} = \frac{\frac{14}{100}}{\frac{99}{100}} = \frac{14}{99}$$

Example: Find the values of $x$ so that $\sum_{n=1}^{\infty}(4x-5)^n$ will converge. Find the sum for those values of x.

$$(4x-5) + (4x-5)^2 + (4x-5)^3 + \cdots$$

$$a \qquad ar \qquad ar^2$$

conv. if $|r|<1$

$a = 4x-5$

$r = 4x-5$

$|4x-5| < 1$

$-1 < 4x-5 < 1$

$4 < 4x < 6$

$\frac{4}{4} < x < \frac{6}{4}$

$1 < x < \frac{3}{2}$

converge for the Interval

$(1, 3/2)$

$$\text{Sum} = \frac{a}{1-r} = \frac{4x-5}{1-(4x-5)} = \boxed{\frac{4x-5}{6-4x}}$$