Section 11.5: Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. The general term, a_n, is of the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ or $a_n = (-1)^{n-1} b_n$, where b_n is a positive number.

The Alternating Series Test (AST): If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{(n-1)} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots.$$

with $b_n > 0$ satisfies:

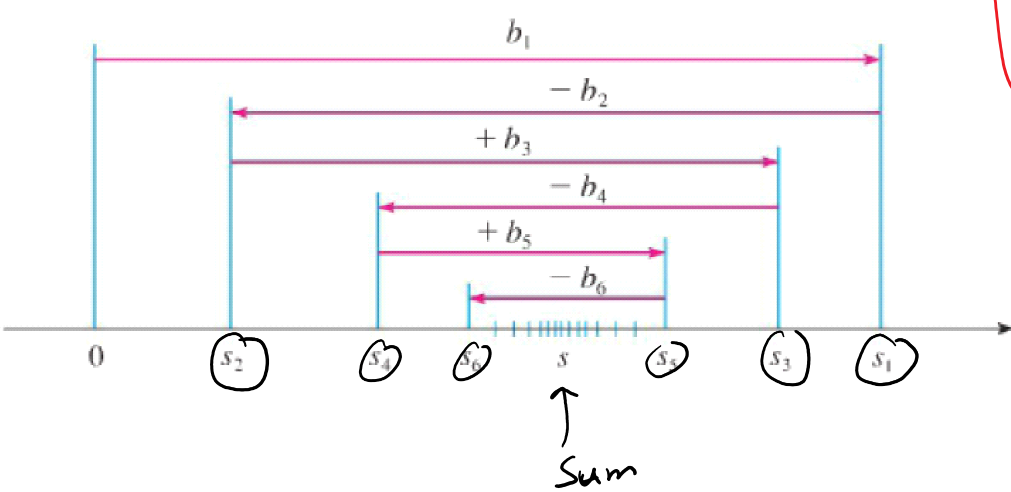
The bn are decreasing

(1) $b_{n+1} \leq b_n$ for all n and (2) $\lim_{n\to\infty} b_n = 0$

Test for div.

then the series is convergent.

AST never Tells us a series will div.



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Example: Does this series converge or diverge? $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$a_n = (-1)^n \frac{1}{n}$

$b_n = \frac{1}{n}$

$b_n = \frac{1}{n}$ $\lim_{n \to \infty} \frac{1}{n} = 0$ ✓

b_n decreasing ✓

by AST This series will converge.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln(2)}$$

Pg 3: Estimation Theorem

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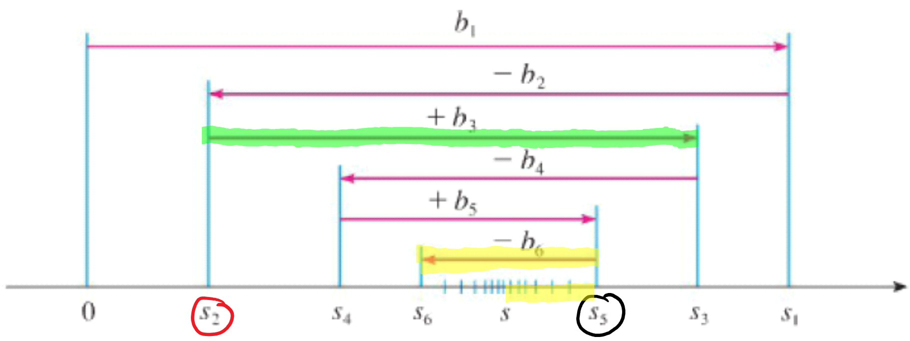
Alternating Series Estimation Theorem: If $s = \sum_{n=1}^{\infty} (-1)^{(n-1)} b_n$ is the sum of an alternating series that satisfies:

(a) $0 < b_{n+1} \le b_n$ and (b) $\lim_{n \to \infty} b_n = 0$

then $|R_n| = |s - s_n| \le b_{n+1}$

error is less than the next b_n



error $= R_2$

error $= R_5$

$S_2 = b_1 - b_2$

$S_5 = b_1 - b_2 + b_3 - b_4 + b_5$

$|R_2| < b_3$

$|R_5| < b_6$

Example: Find a bound on R_4 for the series: $\sum_{n=3}^{\infty} \frac{(-1)^n}{n}$ ← conv. by AST

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n} = \underbrace{\frac{-1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6}}_{S_4} \underbrace{- \frac{1}{7} + \frac{1}{8} \cdots\cdots}_{R_4}$$

$$|R_4| < \frac{1}{7}$$

Example: Do these series converge or diverge?

A) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n^2}\right)$

$b_n = \ln\left(1+\frac{1}{n^2}\right)$

$\lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right) = \ln(1) = 0$ ✓

dec??

$f(x) = \ln(1+x^{-2})$

$f' = \frac{1}{1+x^{-2}} \cdot -2x^{-3} = \frac{\frac{-2}{x^3}}{1+\frac{1}{x^2}} < 0$ for $x>0$

by AST the series will converge.

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B) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n^2} = \sum_{n=1} \frac{(-1)^{n+1} 3^{n+1}}{n^2}$

$\sum -\frac{3^{n+1}}{n^2}$

$b_n = \frac{3^{n+1}}{n^2}$

$\lim_{n\to\infty} \frac{3^{n+1}}{n^2} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^{n+1} \ln(3)}{2n} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^{n+1} \ln(3) \ln(3)}{2}$

$= \infty$

The serie will diverge by $\underbrace{\text{The test for D.V.}}$

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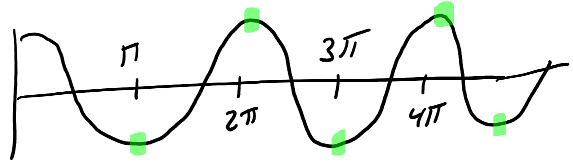
$$(-1)^n \qquad (-1)^{n+1} \qquad (-1)^{n-1}$$

C) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$

$\cos(n\pi)$



$$b_n = \frac{n}{4^n}$$

$$\lim_{n\to\infty} \frac{n}{4^n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{4^n \ln(4)} = 0 \ \checkmark$$

dec?

yes

$$f(x) = \frac{x}{4^x} \qquad f'(x) = \frac{4^x(1) - x\cdot 4^x \ln(4)}{(4^x)^2}$$

$$= \frac{4^x[1 - x\ln(4)]}{(4^x)^2}$$

$$f'(x) = \frac{\boxed{1 - x\ln(4)}}{4^x} < 0$$

$$1 - x\ln(4) = 0$$

$$1 = x\ln(4)$$

$$x = \frac{1}{\ln(4)}$$

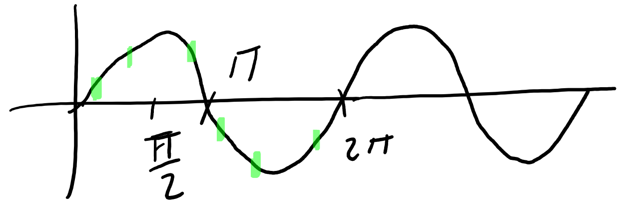
for $x > \frac{1}{\ln(4)}$

The series will converge by AST

D) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2+1}$

None of my test will work on this.

wait until the next section.



3 positive terms then 3 negative terms

not alternating.

Example: Determine if the series converges or diverges. If it converges find a bound for the error of s_6, i.e. R_6

$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$$

$b_n = \sqrt{n+1} - \sqrt{n}$

$$\lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0 \checkmark$$

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dec $f(x) = \sqrt{x+1} - \sqrt{x}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{\sqrt{x} - \sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}} < 0$$

Critical # $x = \frac{1}{2}$

dec ✓

by AST the series will conv.

$$|R_6| < b_7 = \sqrt{7+1} - \sqrt{7} = \sqrt{8} - \sqrt{7}$$

Example: What is the smallest number of terms we must use to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ so that the error is less than $\frac{1}{120}$.

$b_n = \frac{1}{n^2}$

$|R_n| < b_{n+1} < \frac{1}{120}$

$\frac{1}{(n+1)^2} < \frac{1}{120}$

$120 < (n+1)^2$

$\sqrt{120} < n+1$

$\sqrt{120} - 1 < n$

$9.95 < n$

$n = 10$

$n=1$ $b_1 = 1$

$n=2$ $b_2 = \frac{1}{4}$

$n=3$ $b_3 = \frac{1}{9}$

$n=9$ $b_9 = \frac{1}{81}$

$n=10$ $b_{10} = \frac{1}{100}$

$b_{11} = \frac{1}{121}$ "next"

$n = 10$

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Example: What is the minimum number of terms needed so that the sum of this series is correct to 3 decimal places? i.e. error < 0.0005

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!}$$

$b_n = \frac{1}{(2n-1)!}$

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1$

$3! = 3 \cdot 2 \cdot 1$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$b_1 = 1$

$b_2 = \frac{1}{3!} = \frac{1}{6} = .1\overline{6}$

$b_3 = \frac{1}{5!} = .0083$

$b_{n+1} < .0005$

$\frac{1}{(2(n+1)-1)!} < .0005$

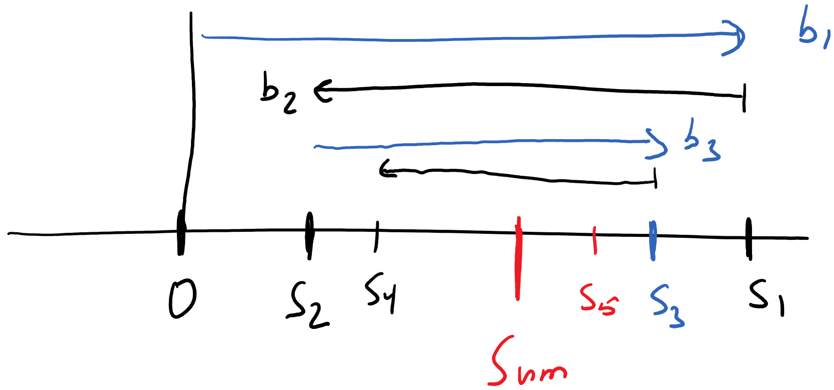
$b_4 = \frac{1}{7!} = .000198$ ← "next term"

need $n = 3$

$b_n > 0$

Example: Given that the alternating series $\sum_{n=1}^{\infty} \underbrace{(-1)^{(n-1)}b_n}$ converges. Is the sum of the first 47 terms, s_{47}, an overestimate or an underestimate for the total sum?

Is the first term pos or neg? positive



overestimate